A Categorification for the Chromatic Polynomial

By

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A Dissertation submitted to
The Faculty of
Columbian College of Arts and Sciences
of The George Washington University in partial satisfaction
of the requirements for the degree of Doctor of Philosophy.

Submitted August 31th, 2005

Dissertation directed by
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Abstract of Dissertation

In recent years, there has been a lot of interests in Khovanov cohomology theory for knots and links. For each link L, Khovanov defines a sequence of cohomology groups whose "graded" Euler characteristic is the Jones polynomial of L. These groups were constructed through a categorification process which starts with a state sum of the Jones polynomial, constructs a group for each term in the summation, and then defines boundary maps between these groups appropriately.

It is natural to ask if similar categorifications can be done for other invariants with state sums. In this thesis, we establish a cohomology theory that categorifies the chromatic polynomial for graphs.

In Chapter (1), we explain how to construct for each graph G a cochain complex whose graded Euler characteristic is the chromatic polynomial of G. This theory is based on the polynomial algebra with one variable X satisfying $X^2 = 0$.

In Chapter (2), we show our cohomology theory satisfies a long exact sequence which can be considered as a categorification for the well-known deletion-contraction rule for the chromatic polynomial. This exact sequence enables us to compute the cohomology groups for several classes of graphs.

This brings some natural questions: Our initial construction was based on the algebra $\mathbb{Z}[X]/(X^2)$. We show in Chapter (3) that it can be extended to a large class of algebras and that some properties carry through.

Another question that arises from the computational examples is to determine which graphs will have torsion in at least one cohomology group. We will answer that in Chapter (4).

Some questions remain open to our cohomology theory. We will state them in Chapter (5).

Acknowledgments

I am deeply indebted to my advisor Yongwu Rong for his time, his patience and his guidance through my research. I came to realize that a good advisor is the key ingredient to a successful dissertation and I think I have been incredibly lucky in this matter.

I wish to thank Joe Bonin for sharing his knowledge about graphs.

I would like to thank Frank Baginski for his help with all non-math related problems.

I would also like to thank all those who made my stay at the George Washington University a pleasant experience for all the nice and useful discussions, the laughs and the time spent together, in particular Fanny Jasso-Hernandez.

And I should probably also thank my husband Pierre Graftieaux who five years ago made me quit my job in France and took me to the United States, circumstances without which I probably would never have started a Ph.D. Discovering both research and the United States turned out to be very fruitful and interesting experiences.

I would like to thank the Columbian College of Arts and Sciences for selecting me for a CCAS Dean's fellowship. It enabled me to focus more fully on my dissertation during the last year of my research.

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Introduction

In recent years, there has been a great deal of interests in Khovanov cohomology[1] theory introduced in [K00]. For each link L in S^3, Khovanov defines a sequence of cohomology groups $\mathcal{H}^{ij}(L)$ whose Euler characteristic $\sum_{i,j}(-1)^i q^j \, rank\mathcal{H}^{i,j}(L)$ is a version of the Jones polynomial of L. These groups were constructed through a categorification process which starts with a state sum of the Jones polynomial, constructs a group for each term in the summation, and then defines boundary maps between these groups appropriately.

The word categorification comes from Khovanov's original paper in which he asked if the quantum invariants of knots and 3-manifolds can be interpreted as Euler Characteristics of some cohomology theories of 3-manifolds. He proved that such an interpretation exists for the Jones polynomial of links in 3-space. Since then, the word categorification has been used to describe the process of interpreting a mathematical object as the (graded) Euler characteristic of a cochain complex. This is its meaning here.

The Khovanov cohomology has proved to be a very powerful tool. First, it is strictly stronger than the Jones polynomial. For instance, it distinguishes knots that the Jones polynomial cannot distinguish [BN02]. Also, there are examples of mutant links with different Khovanov cohomology [W03]. This shows that, contrary to the Jones polynomial, the Khovanov cohomology cannot be defined by a skein relation. Hence the Khovanov cohomology is more than a cosmetic upgrade of the Jones polynomial. Second, it provides a new approach to some results in knot theory. Recently Jacob Rasmussen [R04] defined a new knot invariant based on Khovanov cohomology which he used to derive a new proof of the Milnor conjecture on the slice genus of torus knots. This is the first proof which doesn't depend on the techniques of gauge theory and it is much simpler than all the previously known proofs. Third, there is good evidence for some deep connections with the Ozsvath-Szabo theory which is another recent exciting development in gauge theory type invariant.

We start with a review of Khovanov cohomology for knots and links based on the articles

[1] This construction is often called Khovanov homology rather than Khovanov cohomology. We follow Khovanov (and the usual definition) and call it Khovanov cohomology.

[K00], [BN02], [V02] and [A05]. We opt for a presentation of Khovanov cohomology which is as close as possible to our version for graphs.

For an oriented link L, Mikhail Khovanov [K00] constructed abelian groups $\mathcal{H}^{i,j}(L)$ which depend on two integers i, j such that

$$J(L)(q) = \sum_{i,j} (-1)^i q^j \, rank\left(\mathcal{H}^{i,j}(L)\right)$$

where $J(L)$ is a version of the Jones polynomial and where $rank\left(\mathcal{H}^{i,j}(L)\right)$ denotes the free rank of the abelian group $\mathcal{H}^{i,j}(L)$ which is equal to $\dim_{\mathbb{Q}}(\mathcal{H}^{i,j}(L) \otimes \mathbb{Q})$. These groups were constructed as cohomology groups of cochain complexes according to a process that we describe now.

The *Kauffman bracket* $< D > (q)$ of an unoriented link diagram D is defined by the relations[2]

1. $<$ k disjoint circles $>= (q + q^{-1})^k$
2. $< \quad >=< \quad > -q < \quad >$

where , and stand for link diagrams that are the same except in the neighborhood described in these pictures.

This version of the Kauffman bracket is equal to the original Kauffman bracket multiplied by $(-A^2 - A^{-2})A^{-n}$ where n is the number of crossings of D, after the change of variable $q = -A^{-2}$. This new version of the bracket is not invariant under the second Reidemeister move, as Viro [V04] pointed out. However the invariance is not needed in the construction.

Let $\overrightarrow{D}$ be an oriented link diagram of an oriented link L. Let D be the same diagram without orientation. Let n_+ and n_- be respectively the number of positive and negative crossings of the diagram $\overrightarrow{D}$. Define the *unnormalized Jones polynomial* $J(\overrightarrow{D})$ to be

$$J(\overrightarrow{D}) = (-1)^{-n_-} q^{n_+ - 2n_-} < D >$$

$J(\overrightarrow{D})$ is invariant under ambient isotopy and under the three Reidemeister moves hence it is a link invariant and from now on we will denote it $J(L)(q)$ or simply $J(L)$. This version of the Jones polynomial is equal to the original Jones polynomial multiplied by $-t^{1/2} - t^{-1/2}$ after the change of variable $q = -t^{1/2}$.

We call and the 0-smoothing and 1-smoothing of , respectively. We fix an ordering on the crossings of D and label them 1 to n. With these conventions, each vertex α of the n-dimensional cube $\{0, 1\}^n$ corresponds to a smoothing of all the crossings of D according to α, i.e. if the i^{th} coordinate of α is 0 (resp. 1), the i^{th} crossing is smoothed by a 0-smoothing (resp. a 1-smoothing). The result is a union of planar circles. The diagram D together with a $\alpha \in \{0, 1\}^n$ is called a *state* and is denoted by s_α or simply s if there is no ambiguity. The number of circles produced by the smoothing of D according to α is denoted $|s_\alpha|$ or simply $|s|$. Similarly, we denote by α_s the vertex α corresponding to the state s. The number of 1's in α_s, i.e. the number of 1-smoothings, is denoted by i_s.

The bracket polynomial can be expressed as a state sum

$$< D >= \sum_i (-1)^i \sum_{\text{states } s \text{ s.t. } i_s = i} q^{i_s} (q + q^{-1})^{|s|} \tag{1}$$

[2] Our slightly unorthodox definitions of the Jones polynomial and the Kauffman bracket polynomial follow Khovanov [K00].

which in turn yields a state sum for the unnormalized Jones polynomial,

$$J(\overrightarrow{D}) = (-1)^{-n_-} q^{n_+ - 2n_-} \sum_i (-1)^i \sum_{\text{states } s \text{ s.t. } i_s = i} q^{i_s}(q+q^{-1})^{|s|} \tag{2}$$

The whole procedure of computing the Jones polynomial using this state sum can be depicted as in Figure (1), which describes the case of the right handed trefoil knot. Each column in the diagram corresponds to a value of i and each rectangle describes a state. The polynomial in variable q in the corner of each rectangle is the contribution $q^{i_s}(q+q^{-1})^{|s|}$ of this state to the bracket polynomial.

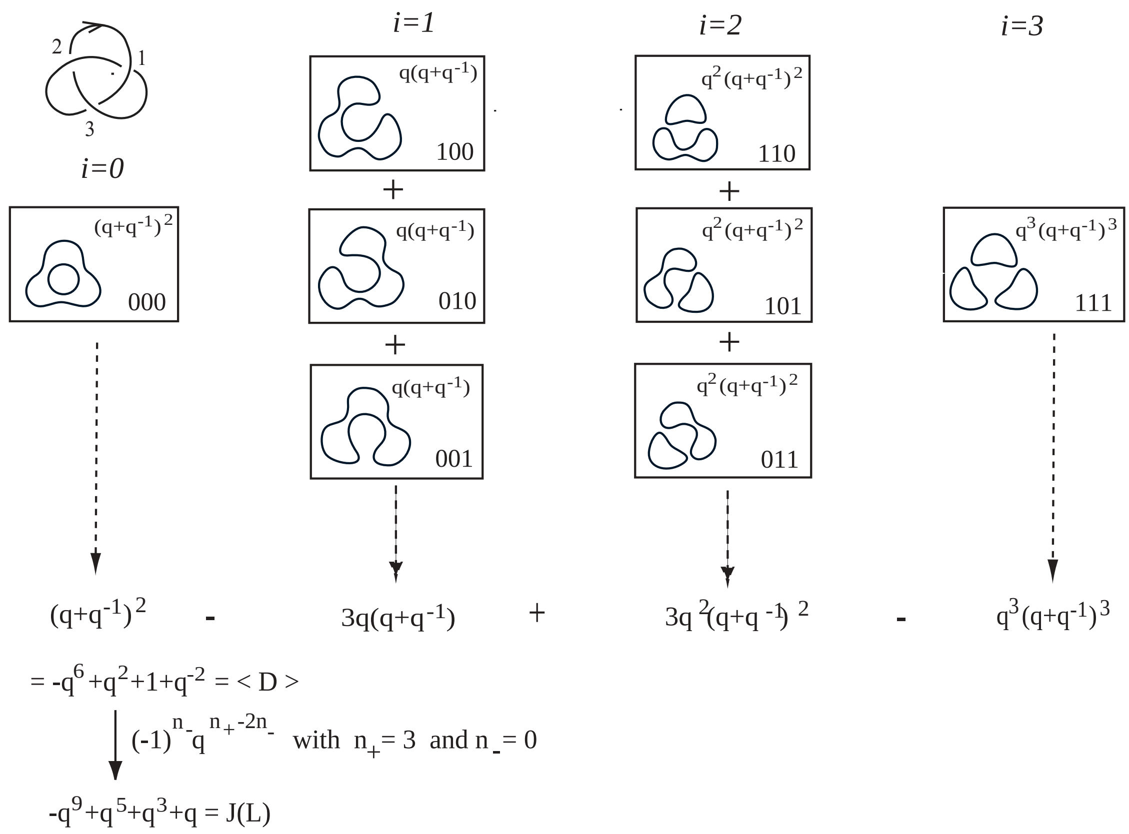


Figure 1: *A state sum based diagram to compute the Jones polynomial of the right handed trefoil knot.*

Before explaining Khovanov's construction, we need to go over some background material.

Graded dimension of a graded $\mathbb{Z}$-module, Graded cochain complex, Graded Euler characteristic: A quick review.

Definition 1. *A $\mathbb{Z}$-module M is said to be* graded *if there exists a direct sum decomposition $M = \oplus_{j=0}^{\infty} M_j$ where each M_j is a $\mathbb{Z}$-submodule. The elements of M_j are called* homogeneous elements of degree j *of M.*

Note that the M_j's are $\mathbb{Z}$-submodules which implies that multiplying by elements of the ring $\mathbb{Z}$ doesn't change the degree. In other words, the elements of the ring $\mathbb{Z}$ have degree 0. The fact that the M_j's are $\mathbb{Z}$-submodules also implies that 0 can be considered to have any degree. This allows torsion.

Definition 2. *Let $j_0 \in \mathbb{N}$ and let $M = \oplus_{j=-j_0}^{\infty} M_j$ be a graded $\mathbb{Z}$-module where M_j denotes the set of homogeneous elements of degree j of M. Assume that each M_j has finite free rank, where the* free rank *of the abelian group M_j is $rank(M_j) := \dim_{\mathbb{Q}}(M_j \otimes \mathbb{Q})$. The* graded dimension of M *is the Laurent series*

$$q \dim M := \sum_{j=-j_0}^{\infty} q^j rank(M_j).$$

Remark 3. *M may have torsion but the graded dimension will not detect it.*

Proposition 4. *Let M and N be graded $\mathbb{Z}$-modules.*

$q \dim (M \oplus N) = q \dim (M) + q \dim (N)$

and $q \dim (M \otimes N) = q \dim (M) \cdot q \dim (N)$

Example 5. *Let $\mathcal{M} = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ be the graded free $\mathbb{Z}$-module with two basis elements v_+ and v_- whose degrees are 1 and -1 respectively. This is the $\mathbb{Z}$-module we will use to construct Khovanov cochain complex.*

Note that $q \dim \mathcal{M} = q + q^{-1}$ and $q \dim \mathcal{M}^{\otimes k} = (q + q^{-1})^k$.

In order to take care of the q^{ℓ} factors in the state sum, we will need an operation on graded $\mathbb{Z}$-modules that shifts the degree of all the elements by ℓ:

Definition 6. *Let $\cdot \{\ell\}$ be the* degree shift *operation on graded $\mathbb{Z}$-modules. That is, if $M = \oplus_j M_j$ is a graded $\mathbb{Z}$-module where M_j denotes the j-th graded component of M, we set $M\{\ell\}_j := M_{j-\ell}$ so that $q \dim M\{\ell\} = q^{\ell} \cdot q \dim M$. In other words, all the degrees are increased by ℓ.*

In order to take care of the $(-1)^{-n_-}$ in the normalization factor $(-1)^{-n_-} q^{n_+ - 2n_-}$ in $J(L)$, we will use the following operation on cochain complexes:

Definition 7. *If $\overline{\mathcal{C}}$ is a cochain complex $\cdots \to \overline{C}^i \to \overline{C}^{i+1} \to \cdots$, we call i the* height *of the cochain group $\overline{C}^i$ of that cochain complex.*

The height shift *operation on cochain complexes, denoted* $\cdot[s]$, *is defined the following way: If* $\mathcal{C} = \overline{\mathcal{C}}[s]$ *is the image of the cochain complex* $\overline{\mathcal{C}}$ *under* $[s]$, *then* $C^i = \overline{C}^{i-s}$ *with all the differentials shifted accordingly.*

Definition 8. *Let* $M = \oplus_j M_j$ *and* $N = \oplus_j N_j$ *be graded* $\mathbb{Z}$*-modules where* M_j *and* N_j *denote the set of homogeneous elements of degree* j *of* M *and* N *respectively. A* $\mathbb{Z}$*-module map* $\phi : M \to N$ *is said to be* graded with degree d *if for all* j, $\phi(M_j) \subseteq N_{j+d}$, *i.e. elements of degree* j *are mapped to elements of degree* $j + d$.
A graded (co)chain complex *is a (co)chain complex for which the cochain groups are graded* $\mathbb{Z}$*-module and the differentials are graded.*

Definition 9. *The* graded Euler characteristic $\chi_q(\mathcal{C})$ of a graded cochain complex $\mathcal{C}$ *is the alternating sum of the graded dimensions of its cohomology groups,*
i.e. $\chi_q(C) = \sum\limits_{0 \leqslant i \leqslant n} (-1)^i \cdot q\dim(H^i)$.

The following was stated in [BN02]. For convenience of the reader, we include a proof here.

Proposition 10. *If the differential is degree preserving and all cochain groups have finite free rank, the graded Euler characteristic is also equal to the alternating sum of the graded dimensions of its cochain groups i.e.*

$$\chi_q(C) = \sum_{0 \leqslant i \leqslant n} (-1)^i q\dim(H^i) = \sum_{0 \leqslant i \leqslant n} (-1)^i q\dim(C^i).$$

Proof. It suffices to show that $\sum\limits_{0 \leq i \leq n} (-1)^i q\dim(H^i) = \sum\limits_{0 \leq i \leq n} (-1)^i q\dim(C^i)$. The corresponding result for the non graded case is well known i.e. for a finite cochain complex $\mathcal{C} = 0 \to C^0 \to C^1 \to ... \to C^n \to 0$ with cohomology groups $H^0, H^1, ..., H^n$, if all the cochain groups are finite dimensional then the Euler characteristic $\chi(\mathcal{C}) = \sum\limits_{0 \leqslant i \leqslant n} (-1)^i rank(H^i)$ is also equal to $\sum\limits_{0 \leqslant i \leqslant n} (-1)^i rank(C^i)$.

Now, let $\mathcal{C}$ be a graded cochain complex with a degree preserving differential. With the above notations, decomposing elements by degree yields $C^i = \bigoplus\limits_{j \geqslant 0} C^{i,j}(G)$. Since the differential is degree preserving, the restriction to elements of degree j, i.e. $0 \to C^{0,j} \to C^{1,j} \to ... \to C^{n,j} \to 0$ is a cochain complex. The previous result tells us $\sum\limits_{0 \leqslant i \leqslant n} (-1)^i rank(H^{i,j})$ $= \sum\limits_{0 \leqslant i \leqslant n} (-1)^i rank(C^{i,j})$. Now, multiply this by q^j and take the sum over all values of j and you get the announced result, since $C^i = \bigoplus\limits_{j \geqslant 0} C^{i,j}(G)$ and $H^i = \bigoplus\limits_{j \geqslant 0} H^{i,j}(G)$. □

We are now ready to explain Khovanov's construction.

The idea

First we produce a graded cochain complex $\overline{\mathcal{C}}(D)$ whose graded Euler characteristic is the bracket polynomial $< D >= \sum_i (-1)^i \sum_{\text{states } s \text{ s.t. } i_s=i} q^{i_s}(q+q^{-1})^{|s|}$.

This seems to be a reasonable goal since this state sum looks like an Euler characteristic since it is an alternating sum. The construction of the cochain complex $\overline{\mathcal{C}}(D)$ is done in two steps: constructing the cochain groups and defining the differential.

Then, simply by shifting the heights and degrees of this cochain complex, we get a second cochain complex $\mathcal{C}$ whose graded Euler characteristic is the Jones polynomial $J(L)$ as announced.

This description of the construction is based on Bar-Natan's presentation of Khovanov cohomology [BN02].

The cochain groups

Let $\overrightarrow{D}$ be an oriented link diagram. Note that the orientation will not be used to construct the cochain complex $\overline{\mathcal{C}}(D)$. This is consistent with the fact that $\overline{\mathcal{C}}(D)$ categorifies the bracket polynomial, an invariant of unoriented links. However, when it comes to deriving the cochain complex $\mathcal{C}(\overrightarrow{D})$ from $\overline{\mathcal{C}}(D)$, the height and degree shifts that we use will depend on the orientation. This is consistent with the fact that $\mathcal{C}(\overrightarrow{D})$ categorifies the Jones polynomial, an invariant of oriented links.

Let $\mathcal{M}$ be as in Example (5). We fix an ordering on the set of crossings of D and label the crossings 1 to n. To each state s labelled by the vertex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of the cube $\{0,1\}^n$, we associate the graded free $\mathbb{Z}$-module $\overline{C}_\alpha(D)$ as follows. We assign a copy of $\mathcal{M}$ to each circle produced by the smoothing of all the crossings of D according to α, we take their tensor product, and then we raise the degrees by i_s. Therefore, $\overline{C}_\alpha(D) = \mathcal{M}^{\otimes |s_\alpha|}\{i_s\}$.

This is shown in Figure (2). The reason for this choice is that the graded dimension of $\overline{C}_\alpha(D)$ is the polynomial $q^{i_s}(q+q^{-1})^{|s_\alpha|}$ that appears in the vertex α of the cube in Figure (1).

To get the cochain groups, we "flatten" the cube by taking direct sums along the columns. A more precise definition is:

Definition 11. *We set the i^{th} cochain group $\overline{C}^i(D)$ of the cochain complex $\overline{\mathcal{C}}(D)$ to be the direct sum of all the $\mathbb{Z}$-modules $\overline{C}_\alpha(D)$ at height i, i.e. $\overline{C}^i(D) = \bigoplus_{|i_\alpha|=i} \overline{C}_\alpha(D)$.*

It remains to define a degree preserving differential for the chain complex $\overline{\mathcal{C}}$.

A degree preserving differential.

So far, in order to define the cochain groups $\overline{C}^i(D)$, we turned each vertex of the cube $\{0,1\}^n$ into a $\mathbb{Z}$-module and then took direct sums along columns. Now, in order to define the differential, we turn each edge of the cube, represented by arrows between vertices in

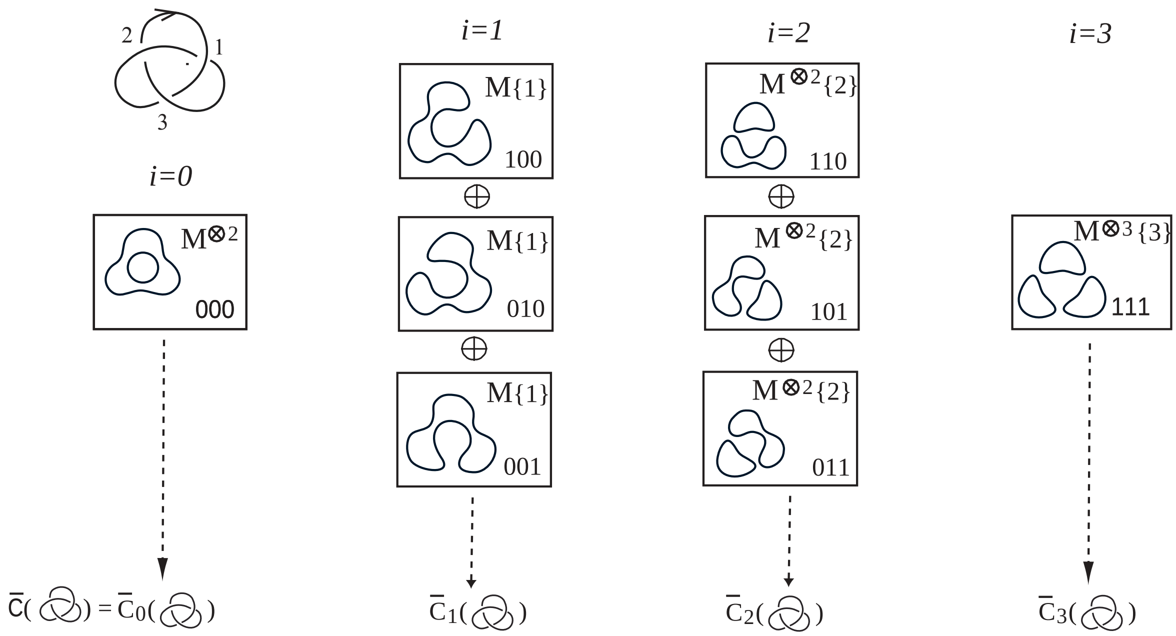


Figure 2: *The cochain groups $\overline{C}^i$ in the case of the right handed trefoil knot.*

Figure (3), into a $\mathbb{Z}$-linear map between $\mathbb{Z}$-modules and then add these maps along columns as shown in Figure (3).

To define the differential maps d^i, we need to make use of the edges of the cube $\{0,1\}^n$. Each edge ξ of $\{0,1\}^E$ can be labelled by a sequence in $\{0,1,*\}^n$ with exactly one $*$. The tail of the edge is obtained by setting $* = 0$ and the head is obtained by setting $* = 1$. The *height* $|\xi|$ is defined to the height of its tail, which is also equal to the number of 1's in ξ.

Given an edge ξ of the cube, let α_1 be its tail and α_2 be its head. The *per-edge* map $d_\xi : \overline{C}_{\alpha_1}(D) \to \overline{C}_{\alpha_2}(D)$ is defined as follows. Changing exactly one marker from 0 to 1 means changing a 0-smoothing to a 1-smoothing. This can either split a circle or join two circles.

⋄ If changing the marker from 0 to 1 joins two circles then we set $d_\xi : \overline{C}_{\alpha_1}(D) \to \overline{C}_{\alpha_2}(D)$ to be identity on the tensor factors corresponding to circles that don't participate and we complete the definition of d_ξ using the $\mathbb{Z}$-linear map $m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ defined on basis elements of $\mathcal{M} \otimes \mathcal{M}$ by

$$m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0 \end{cases}$$

⋄ If changing the marker from 0 to 1 splits a circle then then we set $d_\xi : \overline{C}_{\alpha_1}(D) \to$

$\overline{C}_{\alpha_2}(D)$ to be identity on the tensor factors corresponding to circles that don't participate and we complete the definition of d_ξ using the $\mathbb{Z}$-linear map $\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ defined on basis elements of $\mathcal{M}$ by

$$\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

The cube now commutes because the multiplication map m (resp. the comultiplication map Δ) is associative and commutative (resp. coassociative and cocommutative).

In order for the differential to satisfy $d \circ d = 0$ we want it to anti-commute and this can be achieved by sprinkling negative signs as explained below.

We define the differential $d^i : \overline{C}^i(D) \to \overline{C}^{i+1}(D)$ by $d^i = \sum_{|\xi|=i}(-1)^\xi d_\xi$,where $(-1)^\xi = -1$ (resp. 1) if there is an odd (resp. even) number of 1's before $*$ in ξ. In Figure (3), we have indicated the maps for which $(-1)^\xi = -1$ by a little circle at the tail of the arrow.

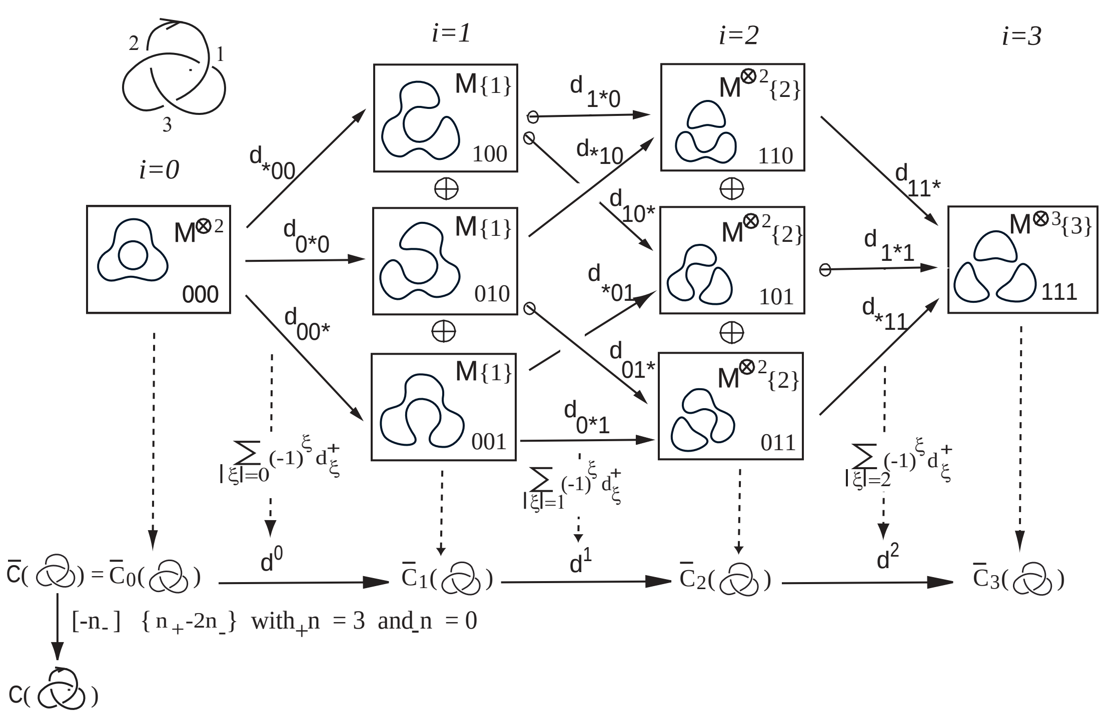


Figure 3: *Per-edge maps and chain maps in the case of the right handed trefoil knot.*

Note that m and Δ are of degree -1 so with the degree shift in the definition of $\overline{C}^i$, the differential is degree preserving.

All this proves that $\overline{\mathcal{C}}(D)$ is indeed a cochain complex with a degree preserving differential. Therefore, $\mathcal{C}(\overrightarrow{D}) = \overline{\mathcal{C}}[-n_-]\{n_+ - 2n_-\}(D)$ is also a cochain complex with a degree preserving differential.

Khovanov's main results

They are summarized in the following theorem:

Theorem 12.

1. *$\mathcal{C}(\overrightarrow{D}) = 0 \to C^0(\overrightarrow{D}) \xrightarrow{d^0} C^1(\overrightarrow{D}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(\overrightarrow{D}) \to 0$ is a graded cochain complex whose differential is degree preserving.*

2. *Although the cochain complex $\mathcal{C}(\overrightarrow{D})$ depends on the choice of a diagram $\overrightarrow{D}$ for the link L, its cohomology groups $\mathcal{H}^i(\overrightarrow{D})$ depend only on the link L. Therefore they are denoted by $\mathcal{H}^i(L)$.*

3. *The graded Euler characteristic of the cochain complex $\mathcal{C}(\overrightarrow{D})$ is the unnormalized Jones polynomial of L:*

$$\chi_q(\mathcal{C}(\overrightarrow{D})) = \sum_{0\leq i\leq n} (-1)^i qdim(\mathcal{H}^i(\overrightarrow{D})) = \sum_{0\leq i\leq n} (-1)^i qdim(C^i(\overrightarrow{D})) = J(L)$$

Proof.

1. has just been proved.

2. The invariance of $\mathcal{H}^i(\overrightarrow{D})$ under Reidemeister moves is a deep result. The proofs can be found in [K00], [BN02] and [A05].

3. We first prove the result for the cochain complex $\overline{C}(D)$. As observed earlier, for each state $q\dim \overline{C}_\alpha(D) = q^{i_s}(q+q^{-1})^{|s_\alpha|}$ by construction. Since $\overline{C}^i(D) = \oplus_{|i_\alpha|=i}\overline{C}_\alpha(D)$, its graded dimension is $q\dim \overline{C}^i(D) = \sum_{|i_\alpha|=i} q^{i_s}(q+q^{-1})^{|s_\alpha|}$.
   Therefore, $\sum_{0\leq i\leq n}(-1)^i q\dim \overline{C}^i(D) = \sum_{0\leq i\leq n}(-1)^i \sum_{|i_\alpha|=i} q^{i_s}(q+q^{-1})^{|s_\alpha|}$, which is equal to $< D >$ by the state sum (1). The differential is degree preserving so by (10), this is also equal to $\sum_{0\leq i\leq n}(-1)^i q\dim \overline{H}^i(D)$.
   Recall that $\mathcal{C}(\overrightarrow{D}) = \overline{\mathcal{C}}(D)[-n_-]\{n_+ - 2n_-\}$. These height and degree shifts are going to multiply the graded Euler characteristic by the normalization factor $(-1)^{-n_-}q^{n_+-2n_-}$. Hence the graded Euler characteristic of $\mathcal{C}(\overrightarrow{D})$ is the unnormalized Jones polynomial as expressed in the state sum (2).

□

The above graded cochain complex can easily be seen to be a bi-graded cochain complex. Let $\mathcal{C}^{i,j}(\overrightarrow{D})$ be the subgroup of $\mathcal{C}^{i}(\overrightarrow{D})$ consisting of homogeneous elements with degree j. Let $d^{i,j}$ be the restriction of d^i to elements with degree j. For each j we have a cochain complex

$$0 \to \mathcal{C}^{0,j}(\overrightarrow{D}) \xrightarrow{d^{0,j}} \mathcal{C}^{1,j}(\overrightarrow{D}) \xrightarrow{d^{1,j}} \cdots \xrightarrow{d^{n-1,j}} \mathcal{C}^{n,j}(\overrightarrow{D}) \to 0$$

The direct sum of these cochain complexes, with the obvious gradings, is equal to the cochain complex $\mathcal{C}(\overrightarrow{D})$. The different gradings don't interfere hence $\mathcal{C}^{i}(\overrightarrow{D}) = \oplus_j \mathcal{C}^{i,j}(\overrightarrow{D})$ and $\mathcal{H}^{i}(\overrightarrow{D}) = \oplus_j \mathcal{H}^{i,j}(\overrightarrow{D})$.

It is natural to ask if similar categorifications can be done for other invariants with state sums, and indeed, several link invariants were categorified, see [BN04], [K03] and [KR04].

In this thesis, we establish a cohomology theory that categorifies the chromatic polynomial for graphs.

In Chapter (1), we explain how to construct for each graph G a cochain complex whose graded Euler characteristic is the chromatic polynomial of G. This theory is based on the polynomial algebra with one variable X satisfying $X^2 = 0$.

In Chapter (2), we show our cohomology theory satisfies a long exact sequence which can be considered as a categorification for the well-known deletion-contraction rule for the chromatic polynomial. This exact sequence enables us to compute the cohomology groups for several classes of graphs.

This brings some natural questions: Our initial construction was based on the algebra $\mathbb{Z}[X]/(X^2)$. We show in Chapter (3) that it can be extended to a large class of algebras and that some properties carry through.

Another question that arises naturally from the computational examples is to determine which graphs will have torsion in at least one cohomology group. We will answer that in Chapter (4).

Some natural questions remain open in our cohomology theory. We will state them in Chapter (5).

Chapter 1

The construction: From a graph to cohomology groups

The results of this section are covered in [HR04].

1.1 Facts about the chromatic polynomial

1.1.1 A brief review for the chromatic polynomial

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each positive integer λ, let $\{1, 2, \cdots, \lambda\}$ be the set of λ-colors. A λ-*coloring* of G is an assignment of a λ-color to vertices of G such that vertices that are connected by an edge in G always have different colors. Let $P_G(\lambda)$ be the number of λ-colorings of G. It is well-known that $P_G(\lambda)$ satisfies the *deletion-contraction relation*

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda)$$

Furthermore, it is obvious that

$$P_{N_n}(\lambda) = \lambda^n \text{ where } N_n \text{ is the graph with } n \text{ vertices and no edges.}$$

These two equations uniquely determines $P_G(\lambda)$. They also imply that $P_G(\lambda)$ is always a polynomial of λ, known as the *chromatic polynomial*.

1.1.2 A state sum for the computation of the chromatic polynomial

As noted earlier, the starting point for a categorification is a state sum formula. There exists such a formula for $P_G(\lambda)$.

For each $s \subseteq E(G)$, let $[G : s]$ be the graph whose vertex set is $V(G)$ and whose edge set is s, and let $k(s)$ be the number of connected components of $[G : s]$. We have

$$P_G(\lambda) = \sum_{s \subseteq E(G)} (-1)^{|s|} \lambda^{k(s)}. \tag{1.1}$$

Equivalently, grouping the terms with the same number of edges yields the state sum formula

$$P_G(\lambda) = \sum_{i \geqslant 0} (-1)^i \sum_{s \subseteq E(G), |s|=i} \lambda^{k(s)}. \tag{1.2}$$

Formula (1.1) follows easily from the well-known state sum formula for the Tutte polynomial $T(G, x, y) = \sum_{s \subseteq E(G)} (x-1)^{r(E)-r(s)}(y-1)^{|s|-r(s)}$, where the rank function $r(s)$ is the number of vertices of G minus the number of connected components of $[G : s]$. One simply applies the known relation $P(G, \lambda) = (-1)^{r(E)} \lambda^{k(G)} T(G; 1-\lambda, 0)$ between the two polynomials.

1.1.3 A diagram for the computation of the chromatic polynomial

Our cochain complex will depend on an ordering of the edges of the graph. Let G be a graph and $E = E(G)$ be the edge set of G. Let $n = |E|$ be the cardinality of E. We fix an ordering on E and denote the edges by $e_1, \cdots, e_n$. For each $s \subseteq E$, the spanning subgraph $[G : s]$ (*spanning* means that it contains all the vertices of G) can be described unambiguously by an element $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of $\{0, 1\}^n$ with the convention $\alpha_i = 1$ if the edge e_i is in s and $\alpha_i = 0$ otherwise. This α is called the *label* of s and will be denoted α_s or simply α. Conversely, to any $\alpha \in \{0, 1\}^n$, we can associate a set s_α of edges of G that corresponds to α. When we think of s in terms of the label α, we may refer to the graph $[G : s]$ as G_α.

The procedure of taking all the spanning subgraphs of G, of computing their number of connected components in order to determine their contribution to the chromatic polynomial as in the state sum of the formula (1.2) can be summarized by a diagram illustrated in Figure (1.1), in which we write $\lambda = 1 + q$ for reasons that will become clear soon.

Each subset of edges s, represented in Figure (1.1) by a labeled rectangle, corresponds to a term in the state sum and therefore will be called a *state*. Equivalently, if we think of the state in term of its label, we might call it a vertex of the cube $\{0, 1\}^n$.

Each state corresponds to a subset s of E, the n-list of $0's$ and $1's$ at the bottom of each rectangle is its label α_s and the term of the form $(1 + q)^{k(s)} = \lambda^{k(s)}$ is its contribution to the chromatic polynomial (without sign).

Note that we have drawn all the states that have the same number of edges in the same column, so that the column with label $i = i_0$ contains all the states with i_0 edges. Such states are called the states of *height* i_0. We denote the height of a state with label α by $|\alpha|$.

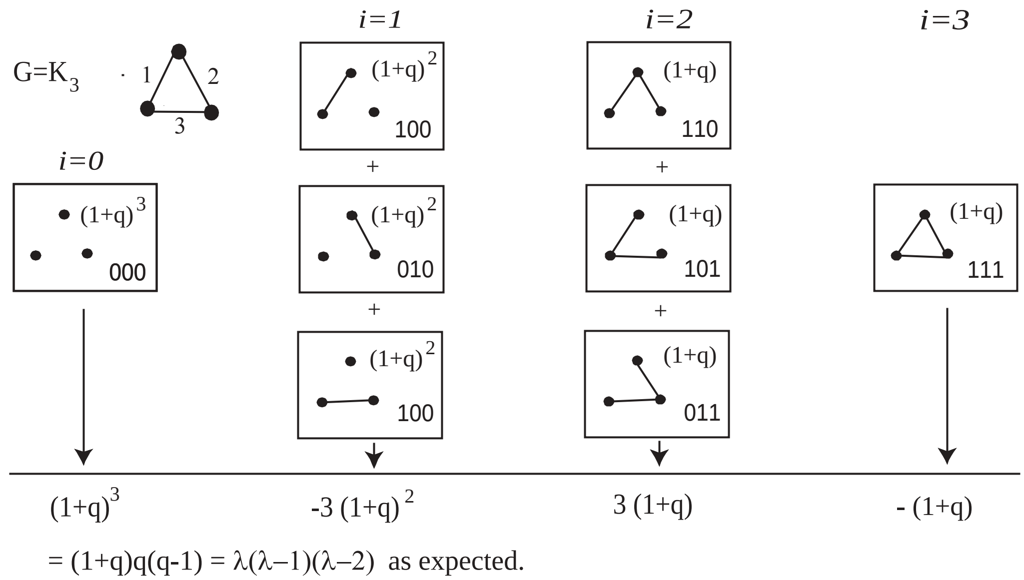


Figure 1.1: *Diagram for a state sum computation of the Chromatic Polynomial when* $G = K_3$

1.2 The cubic complex construction of the cochain complex

1.2.1 The cochain groups

We are going to assign a graded $\mathbb{Z}$-module to each state and define a notion of graded dimension so that the $(1+q)^k$ that appears in the rectangle is the graded dimension of this $\mathbb{Z}$-module.
The construction is inspired by Bar-Natan's description for the Khovanov cohomology for knots and links [BN02].

Example 13. *Let $\mathcal{M}$ be the graded free $\mathbb{Z}$-module with two basis elements 1 and X whose degrees are 0 and 1 respectively. We have $\mathcal{M} = \mathbb{Z}1 \oplus \mathbb{Z}X$ and $q\dim\mathcal{M} = 1+q$. This is the $\mathbb{Z}$-module we will use to construct our cochain complex. Note that it is denoted with a calligraphic $\mathcal{M}$ while we use regular M for a generic $\mathbb{Z}$-module.*
We have $q\dim\left(\mathcal{M}^{\otimes k}\right) = (1+q)^k$.

The cochain groups

We are now ready to explain our construction. Let G be a graph with n ordered edges, and let $\mathcal{M}$ be as in Example (13). To each vertex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of the cube $\{0,1\}^n$, we associate the graded free $\mathbb{Z}$-module $C_\alpha(G) = \mathcal{M}^{\otimes k_\alpha}$ where k_α is the number of components of G_α, by assigning a copy of $\mathcal{M}$ to each connected component and then taking the tensor product. This is shown in Figure (1.2). The reason for this choice is that the polynomial $(1+q)^{k(s)}$ that appears in the vertex α of the cube in Figure (1.1) is $q\dim C_\alpha(G)$ so by substituting this into the state sum formula (1.2) we have

$$P_G(q\dim\mathcal{M}) = \sum_{i\geqslant 0}(-1)^i \sum_{s\subseteq E(G),|s|=i} q\dim C_{\alpha_s}. \tag{1.3}$$

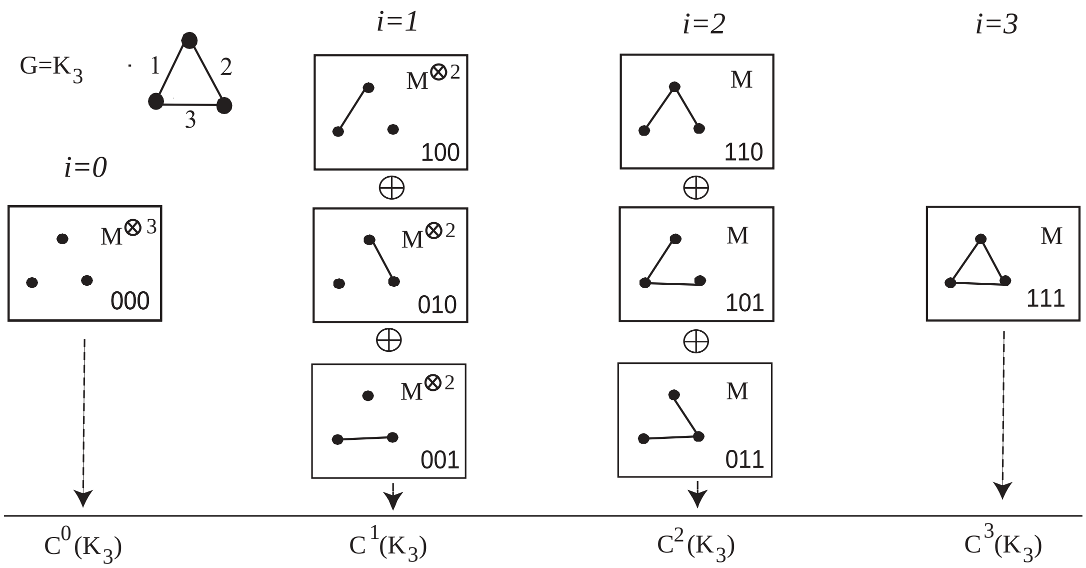
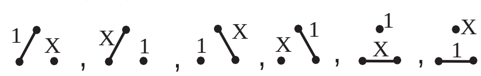


Figure 1.2: *The cochain groups $C^i(K_3)$*

To get the cochain groups, we "flatten" the cube by taking direct sums along the columns. A more precise definition is:

Definition 14. *We set the i^{th} cochain group $C^i(G)$ of the cochain complex $\mathcal{C}(G)$ to be the direct sum of all $\mathbb{Z}$-modules at height i, i.e. $C^i(G) = \bigoplus_{|\alpha|=i} C_\alpha(G)$.*
The grading is given by the degree of the elements and we can write the i^{th} cochain group as $C^i(G) = \bigoplus_{j\geqslant 0} C^{i,j}(G)$ where $C^{i,j}(G)$ denotes the elements of degree j of $C^i(G)$.

For example, the elements of degree one of $C^1(K_3)$ are the linear combinations with coefficients in $\mathbb{Z}$ of the following six elements: 

. These elements form a basis of the free $\mathbb{Z}$-module $C^{1,1}(K_3)$. This will lead to a second description of our cochain complex explained in section (1.3).

Combining the definition of the cochain groups with the fact that the graded dimension of a direct sum is the sum of the graded dimensions, from formula (1.3) we get

$$P_G(q\dim M) = \sum_{i\geqslant 0}(-1)^i q\dim C^i. \tag{1.4}$$

Recall that by Proposition (10), if the differential is degree preserving and all cochain groups have finite free rank, the graded Euler characteristic is also equal to the alternating sum of the graded dimensions of its cochain groups i.e.

$$\chi_q(C) = \sum_{0 \leqslant i \leqslant n} (-1)^i \cdot q\dim(H^i) = \sum_{0 \leqslant i \leqslant n} (-1)^i \cdot q\dim(C^i).$$

In the next paragraph we will attach a degree preserving differential to the groups $C^i(G)$ and thus get a cochain complex $\mathcal{C}(G)$. Using the above result and formula (1.4), we see that by construction, the graded Euler characteristic of this cochain complex will be equal to the chromatic polynomial of the graph G evaluated at the graded dimension of M i.e.

$$P_G(q\dim M) = \sum_{i \geqslant 0} (-1)^i q\dim H^i. \tag{1.5}$$

1.2.2 The differential

Figure (1.3) gives a picture of what the maps will look like and the details can be found right after the figure. The diagram comes first because we thought it might be helpful to have a picture of what is going on while reading the formal definitions.

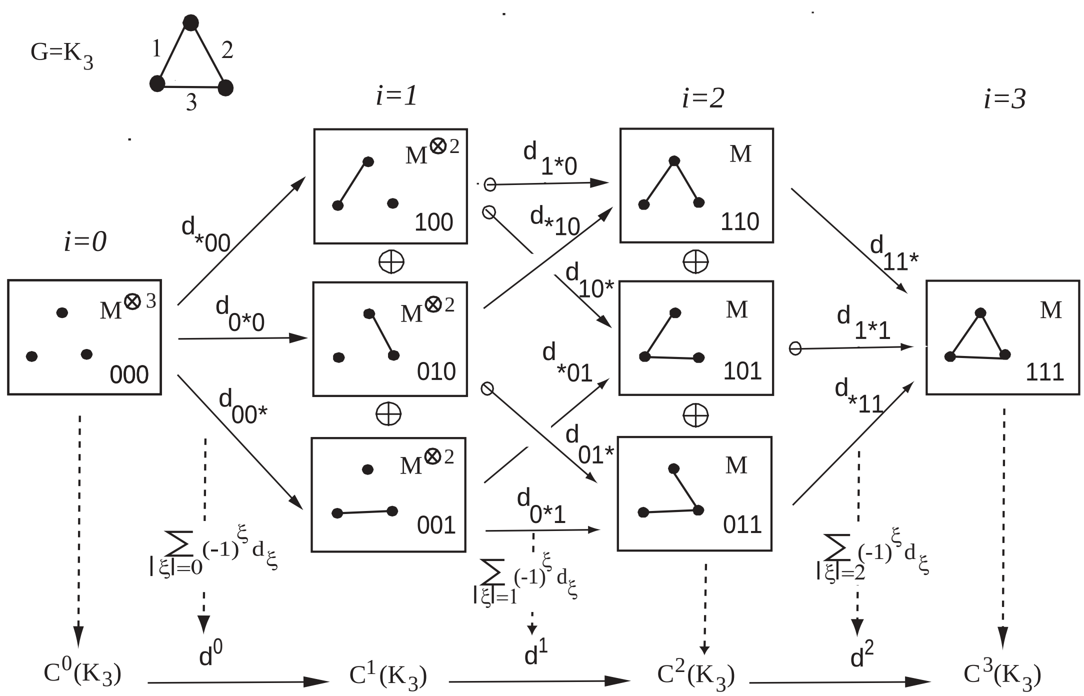


Figure 1.3: *The differentials*

We are first going to define maps between some vertices of the cube $\{0,1\}^n$, called *per-edge maps* since they correspond to edges of the cube. They are represented by labeled arrows in Figure (1.3). We will then build the differential by summing them along columns.

Recall that each vertex of the cube $\{0,1\}^n$ is labeled with some $\alpha = (\alpha_1, ..., \alpha_n) \in \{0,1\}^n$.

Between which vertices are there per-edge maps?

There is a map between two vertices if one can go from one to the other by adding exactly one edge. In other words, there is a map between two vertices if one of the markers α_i is changed from 0 to 1 when you go from the first vertex to the second vertex and all the other α_i are unchanged, and no map otherwise.

Denote by α the label of the first vertex. If the marker which is changed from 0 to 1 has index i_0 then the map will be labeled d_ξ where $\xi = (\xi_1, ..., \xi_n)$ with $\xi_i = \alpha_i$ if $i \neq i_0$ and $\xi_i = *$ if $i = i_0$.

For example, in Figure (1.3), the label $0*1$ of the map d_{0*1} means its domain is the vertex labeled 001 and its target is the vertex labeled 011.

Definition of the per-edge maps

Changing exactly one marker from 0 to 1 corresponds to adding an edge.

⋄ If adding that edge doesn't affect the number of components, then the map is identity on $\mathcal{M}^{\otimes k}$.

⋄ If adding that edge decreases the number of components by one, then we set $d_\xi : \mathcal{M}^{\otimes k} \to \mathcal{M}^{\otimes k-1}$ to be identity on the tensor factors corresponding to components that don't participate and we complete the definition of d_ξ on the affected components using the $\mathbb{Z}$-linear map $m : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ defined on basis elements of $\mathcal{M} \otimes \mathcal{M}$ by

$$m : \begin{cases} m(1 \otimes 1) = 1 \\ m(1 \otimes X) = m(X \otimes 1) = X \\ m(X \otimes X) = 0 \end{cases}$$

Note that identity and m are degree preserving so d_ξ inherits this property.

"Flatten" to get the differential

The differential $d^i : C^i(G) \to C^{i+1}(G)$ of the cochain complex $\mathcal{C}(G)$ is defined by $d^i := \sum_{|\xi|=i} (-1)^\xi d_\xi$ where $|\xi|$ is the number of 1's in ξ and $(-1)^\xi$ is defined in the next paragraph.

Assign a ± 1 factor to the per-edge maps d_ξ

These maps d_ξ make the cube $\{0,1\}^n$ commutative. This is because the multiplication map m is associative and commutative. To get the differential d to satisfy $d \circ d = 0$, it is enough to assign a ± 1 factor to these maps in the following way: Assign -1 to the maps that have an odd number of 1's before the star in their label ξ, and 1 to the others. This is what was denoted $(-1)^\xi$ in the definition of the differential. In Figure (1.3), we have indicated the maps for which $(-1)^\xi = -1$ by a little circle at the tail of the arrow.

A straightforward calculation implies:

Proposition 15. *This defines a differential, that is, $d \circ d = 0$.*

Now, we really have a cochain complex $\mathcal{C}(G)$ where the cochain groups and the differential are defined as in the previous two paragraphs, and as already announced in formula (1.5), we have:

Theorem 16. *The graded Euler characteristic of the cochain complex $\mathcal{C}(G)$ is equal to the chromatic polynomial of the graph G evaluated at the graded dimension of M i.e.*

$$P_G(q \dim \mathcal{M}) = \sum_{i \geqslant 0} (-1)^i q \dim H^i.$$

1.2.3 The cohomology groups are independent of the ordering of the edges

Let G be a graph with edges labeled 1 to n. For any permutation σ of $\{1, .., n\}$, we define G_σ to be the same graph but with edges labeled in the following way: The edge which was labeled k in G is labeled $\sigma(k)$ in G_σ In other words, G is obtained from G_σ by permuting the labels of the edges of G according to σ.

Theorem 17. *The cochain complexes $\mathcal{C}(G)$ and $\mathcal{C}(G_\sigma)$ are isomorphic and therefore, the cohomology groups are isomorphic.*
This implies that the cohomology groups are independent of the ordering of the edges so they are well defined graph invariants.

Proof. Since the group of permutations on n elements is generated by the permutations of the form $(k,\, k+1)$ it is enough to prove the result when $\sigma = (k, k+1)$.

We will define a map f such that the following diagram commutes

$$\begin{array}{llllllllll}
\mathcal{C}(G): & C^0(G) & \stackrel{d^0}{\to} C^1(G) & \stackrel{d^1}{\to} \dots & \stackrel{d^{r-1}}{\to} & C^r(G) & \stackrel{d^r}{\to} \dots & \stackrel{d^{n-1}}{\to} & C^n(G) \\
& \downarrow f & \quad \downarrow f & & & \downarrow f & \quad \dots & & \downarrow f \\
\mathcal{C}(G_\sigma): & C^0(G_\sigma) & \stackrel{d^0}{\to} C^1(G_\sigma) & \stackrel{d^1}{\to} \dots & \stackrel{d^{r-1}}{\to} & C^r(G_\sigma) & \stackrel{d^r}{\to} \dots & \stackrel{d^{n-1}}{\to} & C^n(G_\sigma)
\end{array}$$

and f is an isomorphism. Since $C^i(G)$ is the direct sum of the states with height i, it is enough to define f on each of those states.

For any subset s of E with i edges, there is a state in $C^i(G)$ and one in $C^i(G_\sigma)$ that correspond exactly to those edges. Let $\alpha = (\alpha_1, ..., \alpha_n)$ stand for $\alpha_s(G)$, the label of s in G. The situation is illustrated by the following diagram.

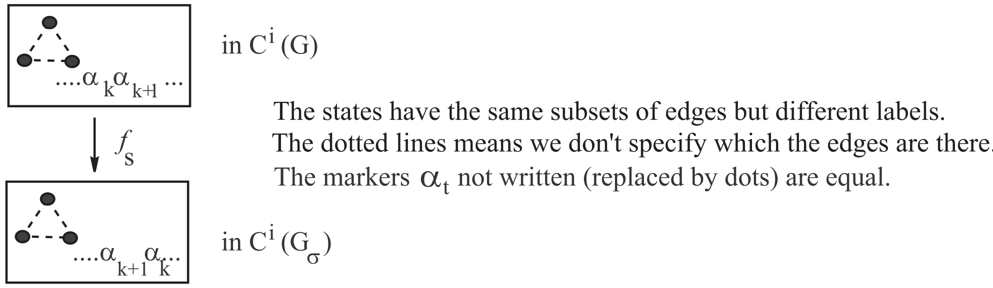


Figure 1.4: *Impact of re-ordering of the edges*

Let f_s be the map between these two states that is equal to $-id$ if $\alpha_k = \alpha_{k+1} = 1$ and equal to id otherwise.

Let $f : C^i(G) \to C^i(G_\sigma)$ be defined by $f = \underset{|s|=i}{\oplus} f_s$.

f is obviously an isomorphism and the fact that the diagram commutes can be checked by looking at the four cases $(\alpha_k, \alpha_{k+1}) = (0,\ 0)$, $(\alpha_k, \alpha_{k+1}) = (1,\ 0)$, $(\alpha_k, \alpha_{k+1}) = (0,\ 1)$ and $(\alpha_k, \alpha_{k+1}) = (1,\ 1)$.

1.2.4 A Poincaré polynomial

Recall that the *Poincaré polynomial* of a sequence of graded $\mathbb{Z}$-modules $\{M^i = \oplus_j M^{i,j}\}_i$ where $M^{i,j}$ is the set of homogeneous elements of M^i of degree j, is defined to be two-variable polynomial (or power series) $P(t,q) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$. With our definition of the graded dimension, this can be rewritten $P(t,q) = \sum_i t^i qdim(M^i)$. The Poincaré polynomial of a sequence of graded $\mathbb{Z}$-modules is a convenient way to store the free ranks of the M^i's.

Following this definition, we define a 2-variable polynomial $R_G(t,q)$ by

$$R_G(t,q) = \sum_{0 \leqslant i \leqslant n} t^i \cdot q\dim\ H^i(G).$$

Proposition 18.
(a) The polynomial $R_G(t,q)$ depends only on the graph.
(b)The chromatic polynomial is a specialization of $R_G(t,q)$ at $t = -1$.

Proof. (a) follows immediately from Theorem (17) and (b) follows from our construction. □

This polynomial is a convenient way to store the information about the free part of the cohomology groups and is, by construction, enough to recover the chromatic polynomial.

1.3 Another Description: The enhanced state construction

These cohomology groups have another description that is similar to Viro's description for the Khovanov cohomology for knots [V02]. We explain the details below.

Let $\{1, X\}$ be a set of *colors*, and $*$ be a product on $\mathbb{Z}[1, X]$ defined by

$$1 * 1 = 1, 1 * X = X * 1 = X \text{ and } \quad X * X = 0$$

Let $G = (V, E)$ be a graph with an ordering on its edges. An *enhanced state* of G is $S = (s, c)$, where $s \subseteq E$ and c is an assignment of 1 or X to each connected component of the spanning subgraph $[G : s]$. For each enhanced state S, define

$$i(S) = \# \text{ of edges in } s, \text{ and } j(S) = \# \text{ of } X \text{ in } c.$$

Note that $i(S)$ depends only on the underlying state s, not on the color assignment that makes it an enhanced state, so we may write it $i(s)$.

▲ Let $C^{i,j}(G) := Span\{S \mid S \text{ is an enhanced state of } G \text{ with } i(S) = i, j(S) = j\}$, where the span is taken over $\mathbb{Z}$.

▲ We define the differential

$$d : C^{i,j}(G) \to C^{i+1,j}(G)$$

as follows. For each enhanced state $S = (s, c)$ in $C^{i,j}(G)$, define $d(S) \in C^{i+1,j}(G)$ by

$$d(S) = \sum_{e \in E(G) - s} (-1)^{n(e)} S_e$$

where $n(e)$ is the number of edges in s that are ordered before e, S_e is an enhanced state or 0 defined as follows. Let $s_e = s \cup \{e\}$. Let $E_1, \cdots, E_k$ be the components of $[G : s]$. If e connects some E_i, say E_1, to itself, then the components of $[G : (s \cup \{e\})]$ are $E_1 \cup \{e\}, E_2, \cdots, E_k$. We define $c_e(E_1 \cup \{e\}) = c(E_1), c_e(E_2) = c(E_2), \cdots, c_e(E_k) = c(E_k)$, and S_e is the enhanced state (s_e, c_e). If e connects some E_i to E_j, say E_1 to E_2, then the components of $[G : s_e]$ are $E_1 \cup E_2 \cup \{e\}, E_3, \cdots, E_k$. We define $c_e(E_1 \cup E_2 \cup \{e\}) = c(E_1) * c(E_2), c_e(E_3) = c(E_3), \cdots, c_e(E_k) = c(E_k)$.

Note that if $c(E_1) = c(E_2) = X$, $c_e(E_1 \cup E_2 \cup \{e\}) = X * X = 0$, and therefore c_e is not considered as a coloring. In this case, we let $S_e = 0$. In all other cases, c_e is a coloring and we let S_e be the enhanced state (s_e, c_e).

One may find it helpful to think of d as the operation that adds each edge not in s, adjusts the coloring using $*$, and then sums up the states using appropriate signs. In the case when an illegal color of 0 appears, due to the product $X * X = 0$, the contribution from that edge is counted as 0.

1.4 This construction is equivalent to the cubic complex construction.

At first sight, the two constructions look different because the cubic complex construction yields only one cochain complex whereas the enhanced states construction gives rise to a sequence of cochain complexes, one for each degree j. This can be easily solved by splitting the cochain complex of the cubic complex construction into a sequence of cochain complexes, one for each degree j. More precisely, let $\mathcal{C} = 0 \to C^0 \to C^1 \to ... \to C^n \to 0$ be a graded cochain complex with a degree preserving differential. Decomposing elements of each cochain group by degree yields $C^i = \bigoplus_{j\geqslant 0} C^{i,j}$. Since the differential is degree preserving, the restriction to elements of degree j, i.e. $0 \to C^{0,j} \to C^{1,j} \to ... \to C^{n,j} \to 0$ is a cochain complex denoted by $\mathcal{C}^j$. It is clear that $\mathcal{C}$ is the direct sum of these cochain complexes.

We are now ready to see that for a fixed j, the cochain complexes obtained via the two construction are isomorphic. For this section, denote the one obtained via the cubic complex construction by $\mathcal{C}^j$ and the one obtained via the enhanced state construction by $\widetilde{\mathcal{C}}^j$.

Both cochain complexes have free cochain groups so it is enough to define the chain map on basis elements. We will associate each enhanced state $S = (s, c)$ of $\widetilde{C}^{i,j}(G)$ to an unique basis element in $C^{i,j}(G)$ and show that this defines an isomorphism of cochain complexes. First, $s \subseteq E(G)$ naturally corresponds to the vertex $\alpha = (\alpha_1, \cdots, \alpha_n)$ of the cube, where $\alpha_k = 1$ if $e_k \in s$ and $\alpha_k = 0$ otherwise. The corresponding $\mathbb{Z}$-module $C_\alpha(G)$ is obtained by assigning a copy of $\mathcal{M}$ to each connected component of $[G, s]$ and then taking tensor product. The color c naturally corresponds to the basis element $x_1 \otimes \cdots \otimes x_k$ where x_ℓ is the color associated to the ℓ-th component of $[G : s]$.

It is not difficult to see that this defines an isomorphism on the cochain group that commutes with the differentials. Therefore, the two complexes are isomorphic.

Chapter 2

Properties

In this section, we demonstrate some properties of our cohomology theory, as well as some computational examples. The results of this section are covered in [HR04].

2.1 An Exact Sequence

The chromatic polynomial satisfies a well-known deletion-contraction rule: $P(G,\lambda) = P(G-e,\lambda) - P(G/e,\lambda)$. Here we show that our cohomology groups satisfy a naturally constructed long exact sequence involving $G, G-e$, and G/e. Furthermore, by taking the graded Euler-characteristic of the long exact sequence, we recover the deletion-contraction rule. Thus our long exact sequence can be considered as a "categorification" of the deletion-contraction rule.

▲ We explain the exact sequence in terms of the enhanced state sum approach. Let G be a graph and e be an edge of G. We order the edges of G so that e is the last edge. This induces natural orderings on G/e and on $G-e$ by deleting e from the list. We define homomorphisms $\alpha_{ij} : C^{i-1,j}(G/e) \to C^{i,j}(G)$ and $\beta_{ij} : C^{i,j}(G) \to C^{i,j}(G-e)$. These two maps will be abbreviated by α and β from now on. Let v_e and w_e be the two vertices in G connected by e. Intuitively, α expands v_e by adding e, and β is the projection map. We explain more details here.

▲ First, given an enhanced state $S = (s, c)$ of G/e, let $\widetilde{s} = s \cup \{e\}$. The number of components of $[G/e : s]$ and $[G : \widetilde{s}]$ are the same. In fact, the components of $[G/e : s]$ and the components of $[G : \widetilde{s}]$ are the same except the one containing v_e where v_e in G/e is replaced by e in G. Thus, c automatically yields a coloring of components of $[G : \widetilde{s}]$, which we denote by $\widetilde{c}$. Let $\alpha(S) = (\widetilde{s}, \widetilde{c})$. It is an enhanced state in $C^{i,j}(G)$. Extend α linearly and we obtain a homomorphism $\alpha : C^{i-1,j}(G/e) \to C^{i,j}(G)$.

▲ Next, we define the map $\beta : C^{i,j}(G) \to C^{i,j}(G-e)$. Let $S = (s, c)$ be an enhanced state of G. If $e \notin s$, S is automatically an enhanced state of $G-e$ and we define $\beta(S) = S$. If $e \in s$, we define $\beta(S) = 0$. Again, we extend β linearly to obtain the map $\beta : C^{i,j}(G) \to C^{i,j}(G-e)$.

One can sum up over j, and denote the maps by $\alpha_i : C^{i-1}(G/e) \to C^i(G)$ and $\beta_i : C^i(G) \to C^i(G-e)$. Again, they will be abbreviated by α and β. Both are degree preserving maps since the index j is preserved.

Lemma 19. *α and β are chain maps such that $0 \to C^{i-1,j}(G/e) \xrightarrow{\alpha} C^{i,j}(G) \xrightarrow{\beta} C^{i,j}(G-e) \to 0$ is a short exact sequence.*

Proof. ▲ First we show that α is a chain map. That is,

$$\begin{array}{ccc} C^{i-1,j}(G/e) & \xrightarrow{\alpha} & C^{i,j}(G) \\ \downarrow d_{G/e} & & \downarrow d_G \\ C^{i,j}(G/e) & \xrightarrow{\alpha} & C^{i+1,j}(G) \end{array}$$

commutes. Let (s, c) be an enhanced state of G/e, we have

$$d_G \circ \alpha((s,c)) = d_G(s \cup \{e\}, \widetilde{c}) = \sum_{e_k \in E(G)-(s\cup\{e\})} (-1)^{n_G(e_k)} (s \cup \{e, e_k\}, (\widetilde{c})_{e_k})$$

where $n_G(e_k)$ is the number of edges in $s \cup \{e\}$ that are ordered before e_k in G.

We also have $\alpha \circ d_{G/e}((s,c)) = \alpha\left(\sum_{e_k \in E(G/e)-s} (-1)^{n_{G/e}(e_k)} (s \cup \{e_k\}, c_{e_k})\right)$
$= \sum_{e_k \in E(G/e)-s} (-1)^{n_{G/e}(e_k)} (s \cup \{e_k, e\}, \widetilde{(c_{e_k})})$ where $n_{G/e}(e_k)$ is the number of edges in s that are ordered before e_k in G/e.

The two summations contain the same list of e_k's since $E(G) - (s \cup \{e\}) = E(G/e) - s$. It is also easy to see that $(\widetilde{c})_{e_k} = \widetilde{(c_{e_k})}$. Finally, $n_G(e_k) = n_{G/e}(e_k)$ since e is ordered last. It follows that $d_G \circ \alpha = \alpha \circ d_{G/e}$ and therefore α is a chain map.

▲ Next, we show that β is a chain map by proving the commutativity of

$$\begin{array}{ccc} C^{i,j}(G) & \xrightarrow{\beta} & C^{i,j}(G-e) \\ \downarrow d_G & & \downarrow d_{G-e} \\ C^{i+1,j}(G) & \xrightarrow{\beta} & C^{i+1,j}(G-e) \end{array}$$

Let $S = (s, c)$ be an enhanced state of G.

△ If $e \in s$, we have $\beta(S) = 0$ and thus $d_{G-e} \circ \beta(S) = 0$. We also have $d_G(S) = \sum_{e_k \in E(G)-s} (-1)^{n_G(e_k)} (s \cup \{e_k\}, c_{e_k})$. Since $e \in s \cup \{e_k\}$, $\beta \circ d_G(S) = 0$.

△ If $e \notin s$, we have $d_{G-e} \circ \beta(S) = d_{G-e}(S) = \sum_{e_k \in E(G-e)-s} (-1)^{n_{G-e}(e_k)} (s \cup \{e_k\}, c_{e_k})$. We also have $d_G(S) = \sum_{e_k \in E(G)-s} (-1)^{n_G(e_k)} (s \cup \{e_k\}, c_{e_k}) = S_1 + S_2$, where $S_1 = \sum_{e_k \in E(G)-(s\cup\{e\})} (-1)^{n_G(e_k)} (s \cup \{e_k\}, c_{e_k})$ corresponds to the terms with $e_k \neq e$, and $S_2 = (-1)^{n_G(e_k)} (s \cup \{e\}, c_e)$ corresponds to the term $e_k = e$. By our definition of β, $\beta(S_1) = S_1, \beta(S_2) = 0$.

Finally, $n_G(e_k) = n_{G-e}(e_k)$ since e is ordered last, and it follows that $d_{G-e}\circ\beta(S) = \beta\circ d_G(S)$ as well in this case.

▲ Next, we prove the exactness. Each element in $C^{i-1,j}(G/e)$ can be written as $x = \sum n_k(s_k, c_k)$ where $n_k \neq 0$ and (s_k, c_k)'s are pairwise distinct enhanced states of G/e. It is not hard to see that $(\tilde{s_k}, \tilde{c_k})$'s are pairwise distinct enhanced states of G. Thus $\alpha(x) = \sum n_k(\tilde{s_k}, \tilde{c_k}) \neq 0$ in $C^{i-1,j}(G)$. Hence $\ker\alpha = 0$. Next, $\mathrm{Im}\alpha = \mathrm{Span}\{(s,c) | (s,c)$ is an enhanced state of G and $e \in s\} = \ker\beta$. Finally, β is a projection map that maps onto $C^{i,j}(G-e)$. □

The Zig-Zag lemma in homological algebra implies :

Theorem 20. *Given a graph G and an edge e of G, for each j there is a long exact sequence*

$0 \to H^{0,j}(G) \xrightarrow{\beta^*} H^{0,j}(G-e) \xrightarrow{\gamma^*} H^{0,j}(G/e) \xrightarrow{\alpha^*} H^{1,j}(G) \xrightarrow{\beta^*} H^{1,j}(G-e) \xrightarrow{\gamma^*} H^{1,j}(G/e) \to \ldots \to H^{i,j}(G) \xrightarrow{\beta^*} H^{i,j}(G-e) \xrightarrow{\gamma^*} H^{i,j}(G/e) \xrightarrow{\alpha^*} H^{i+1,j}(G) \to \ldots$

If we sum over j, we have a degree preserving long exact sequence:

$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(G/e) \to \ldots \to H^i(G) \xrightarrow{\beta^*} H^i(G-e) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} H^{i+1}(G) \to \ldots$

Remark 21. *It is useful to understand how the maps $\alpha^*, \beta^*, \gamma^*$ act in an intuitive way. The descriptions for α^* and β^* follow directly from our construction: α^* expands the edge e, β^* is the projection map. The description for γ^*, the connecting homomorphism, follows from the standard diagram chasing argument in the zig-zag lemma and the result is as follows. For each cycle z in $C^{i,j}(G-e)$ represented by the chain $\sum n_k(s_k, c_k)$, $\gamma^*(z)$ is represented by the chain $(-1)^i \sum n_k(s_k \cup \{e\}/e, (c_k)_e)$, where $s_k \cup \{e\}/e$ is the subset of $E(G/e)$ obtained by adding e to s_k and then contracting e to v_e, $(c_k)_e$ is the coloring defined in section 1.3.*

Remark 22. *It can sometimes be convenient to allow negative heights for cohomology groups with the conventions that for any graph G, $H^i(G) = 0$ if $i < 0$ and α^*, β^* and $\gamma^* = 0$ when their domain has negative height. Note that the previous exact sequence remains exact of we add these groups with negative heights. The degree preserving long exact sequence becomes*

$\cdots \to H^{-1}(G) \xrightarrow{\beta^*} H^{-1}(G-e) \xrightarrow{\gamma^*} H^{-1}(G/e) \xrightarrow{\alpha^*} H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(G/e) \to \ldots \to H^i(G) \xrightarrow{\beta^*} H^i(G-e) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} H^{i+1}(G) \to \ldots$

▲ We now check that by taking the graded Euler-characteristic of the long exact sequence, we recover the deletion-contraction rule as announced. Thus our long exact sequence can be considered as a categorification of the deletion-contraction rule.

The exact sequence for (G, e) is:

$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} H^1(G-e) \xrightarrow{\gamma^*} H^1(G/e) \to \ldots \to H^i(G) \xrightarrow{\beta^*} H^i(G-e) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} H^{i+1}(G) \to \ldots$

First note that since the sequence is exact, its graded Euler-characteristic is zero. This can be seen because the graded Euler characteristic is the sum of the (regular) Euler characteristics in each degree j with a factor q^j and we know that the (regular) Euler-characteristic of an exact sequence is zero.

Now, if we take the graded Euler-characteristic of the exact sequence for (G, e), we get

$q \dim H^0(G) - q \dim H^0(G-e) + q \dim H^0(G/e) - q \dim H^1(G) + q \dim H^1(G-e) + \cdots + (-1)^i q \dim H^i(G) + (-1)^{i+1} q \dim H^i(G-e) + (-1)^i q \dim H^i(G/e) + \cdots = 0$

Hence, $\sum_i (-1)^i q \dim H^i(G) + \sum_i (-1)^{i+1} q \dim H^i(G-e) + \sum_i (-1)^i q \dim H^i(G/e) = 0$ i.e. $P_G(1+q) - P_{G-e}(1+q) + P_{G/e}(1+q) = 0$, which is the deletion-contraction rule for the chromatic polynomial.

A detailed example showing how to compute the cohomology groups of the complete graph on three vertices K_3 using the exact sequence is provided in Section (6.1).

2.2 Graphs with loops

Proposition 23. *If the graph has a loop then all the cohomology groups are trivial.*

Proof. Let G be a graph with a loop ℓ. The exact sequence for (G, ℓ) is

$0 \to H^0(G) \to H^0(G-\ell) \xrightarrow{\gamma^*} H^0(G/\ell) \to H^1(G) \to H^1(G-\ell) \xrightarrow{\gamma^*} H^1(G/\ell) \to \ldots \to H^i(G) \to H^i(G-\ell) \xrightarrow{\gamma^*} H^i(G/\ell) \to H^{i+1}(G) \to \ldots$

Using our description of the connecting homomorphism γ^* in Remark (21), we get that the map $H^i(G-\ell) \xrightarrow{\gamma^*} H^i(G/\ell)$ is $(-1)^i id$. Therefore, $H^i(G) = 0$ for all i. □

2.3 Graphs with multiple edges

Proposition 24. *The cohomology group are unchanged if all the multiple edges of a graph are replaced by single edges.*

Proof. Assume that in some graph G the edges e_1 and e_2 connect the same vertices. In G/e_2, e_1 becomes a loop so as observed earlier, $H^i(G/e_2) = 0$ for all i. It follows from the long exact sequence that $H^i(G-e_2)$ and $H^i(G)$ are isomorphic groups. One can repeat the process until there is at most one edge connecting two given vertices without changing the cohomology groups. □

2.4 Cohomology groups of the disjoint union of two graphs

Let G_1 and G_2 be two graphs and consider their disjoint union $G_1 \sqcup G_2$. On the cochain complex level, we have $\mathcal{C}(G_1 \sqcup G_2) = \mathcal{C}(G_1) \otimes \mathcal{C}(G_2)$.

Theorem 25. *For each $i \in \mathbb{N}$, we have :*

$$H^i(G_1 \sqcup G_2) \cong \left[\bigoplus_{p+q=i} H^p(G_1) \otimes H^q(G_2) \right] \oplus \left[\bigoplus_{p+q=i+1} H^p(G_1) * H^q(G_2) \right]$$

*where * denotes the torsion product of two abelian groups.*
If we decompose the groups by degree, we get that for each $k, i \in \mathbb{N}$, we have :

$$H^{i,j}(G_1 \sqcup G_2) \cong \left[\bigoplus_{\substack{p+q=i \\ s+t=j}} H^{p,s}(G_1) \otimes H^{q,t}(G_2) \right] \oplus \left[\bigoplus_{\substack{p+q=i+1 \\ s+t=j}} H^{p,s}(G_1) * H^{q,t}(G_2) \right]$$

Proof. This is a corollary of Künneth's theorem, since the chain complexes $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ are free.

Details about the Künneth's theorem and the torsion product can be found in J. Munkres's algebraic topology book [M84], on pages 342 and 327 respectively. Basically, the Künneth's theorem tells you that some groups built from the cohomology groups via direct sums and tensor products are related by an exact sequence and that this sequence splits when the chains complexes are free. ☐

Corollary 26. *The Poincaré polynomials are multiplicative under disjoint union i.e.* $R_{G_1 \sqcup G_2}(t, q) = R_{G_1}(t, q) \cdot R_{G_2}(t, q)$

Theorem (25) also implies

Example 27. *Disjoint union with the one vertex graph:* $H^k(G \sqcup \bullet) \cong H^k(G) \otimes (\mathbb{Z} \oplus \mathbb{Z}X)$.

2.5 Adding or contracting a pendant edge

An edge in a graph is called a *pendant* edge if the degree of one of its endpoints is one. Let G be a graph and e be a pendant edge of G. Let G/e be the graph obtained by contracting e to a point. We will study the relation between the cohomology groups of G and G/e.

Recall that the notation $\{.\}$ denotes the degree shift in graded $\mathbb{Z}$-modules. Its meaning is that, for a given graded $\mathbb{Z}$ -module M, $M\{\ell\}$ denotes the $\mathbb{Z}$-module isomorphic to M with the degree of each homogeneous element being shifted up by ℓ. For example, $\mathbb{Z}\{1\}$ denotes an abelian group generated by one element of degree one and $\mathbb{Z}^3\{2\}$ denotes a rank three free abelian group whose elements are of degree two.

We have

Theorem 28. *Let e be a pendant edge in a graph G. G can be represented by S, where the S stands for "something". For each i, $H^i($ S $) \cong H^i($ S $)\{1\} \cong H^i($ S $)\otimes\mathbb{Z}\{1\}$.*

Proof. Consider the operations of contracting and deleting e in G. The graph G/e is S and $G - e$ is S where the isolated vertex is denoted v. The exact sequence on (G, e) is $0 \to H^0($ S $) \xrightarrow{\beta^*} H^0($ S $) \xrightarrow{\gamma^*} H^0($ S $) \xrightarrow{\alpha^*} \cdots \xrightarrow{\alpha^*} H^i($ S $) \xrightarrow{\beta^*} H^i($ S $) \xrightarrow{\gamma^*} H^i($ S $) \xrightarrow{\alpha^*} \cdots$

Thus we need to understand the map

$$H^i(\text{ S }) \xrightarrow{\gamma^*} H^i(\text{ S })$$

By Theorem (25),

$$H^i(\text{ S }) \cong H^i(\text{ S }) \otimes [\mathbb{Z}\oplus\mathbb{Z}\{1\}] \cong H^i(\text{ S }) \oplus \left[H^i(\text{ S }) \otimes \mathbb{Z}\{1\}\right]$$

by a natural isomorphism h_*, which is induced by the isomorphism h described as follows.

$$h : C^i(\text{ S }) \to C^i(\text{ S }) \oplus \left[C^i(\text{ S }) \otimes \mathbb{Z}\{1\}\right]$$

Each enhanced state S in $C^i($ S $)$ either assigns the color 1 or the color X to v. If it assigns 1 to v, h sends S to $(S_1, 0)$ where S_1 is the "restriction" of S to S. If it assigns X to v, h sends S to $(0, S_1 \otimes q)$. This extends to a degree preserving isomorphism on cochain groups and induces the isomorphism h_* on cohomology groups.

We therefore will identify $H^i($ S $)$ with $H^i($ S $) \oplus \left[H^i(\text{ S }) \otimes \mathbb{Z}\{1\}\right]$.

Claim: $(-1)^i\gamma^* \circ (h_*)^{-1} : H^i($ S $) \oplus \left[H^i(\text{ S }) \otimes \mathbb{Z}\{1\}\right] \to H^i($ S $)$ satisfies $(-1)^i\gamma^* \circ (h_*)^{-1}(x, 0) = x$ for all $x \in H^i($ S $)$.

Proof of Claim.

Let x be in $H^i($ S $)$. x is the equivalence class of a sum of terms of the form (s, c) in S. Under the map $(h_*)^{-1}$, each of these terms is "extended" to be an element in $C^i($ S $)$ by adding v to $\left[\text{ S } : s\right]$ and assigning it the color 1.

The map γ^* is described in Remark (21). Let y be in $H^i($ S $)$. y is the equivalence class of a sum of terms of the form (s, c) in $H^i($ S $)$. Basically, for each each (s, c), $(-1)^i\gamma^*$ adds the edge e, adjusts the colorings, then contracts e to a point. Hence applying $(h_*)^{-1}$ then $(-1)^i\gamma^*$ yields the original graph S. The color for each state in x remains the same since v is colored by 1 and multiplication by 1 is the identity map. This proves that $(-1)^i\gamma^* \circ (h_*)^{-1}(x, 0) = x$.

The claim implies that γ^* is onto for each i. Thus the above long exact sequence becomes a collection of short exact sequences.

$0 \to H^i($$) \xrightarrow{\beta^*} H^i($$) \xrightarrow{\gamma^*} H^i($$) \to 0$ (2.1)

After passing to the isomorphism h_*, the exact sequence becomes

$0 \to H^i($$) \longrightarrow H^i($$) \oplus \Big[H^i($$) \otimes \mathbb{Z}\{1\}\Big] \longrightarrow H^i($$) \to 0$

The next lemma implies that $H^i($$) \cong H^i($$) \otimes \mathbb{Z}\{1\}$. $\square$

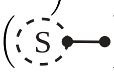
Lemma 29. *Let A and B be graded abelian groups, and $p : A \oplus B \to A$ be a degree preserving projection with $p(a, 0) = a$ for all $a \in A$. Then $\ker p \cong B$ via a degree preserving isomorphism.*

Proof. For each $b \in B$, let $a_b = p(0, b) \in A$. Then $p(-a_b, b) = 0$ and therefore $(-a_b, b) \in \ker p$. Define $f(b) = (-a_b, b)$. It is a standard exercise to verify that f is a degree preserving isomorphism from B to $\ker p$. $\square$

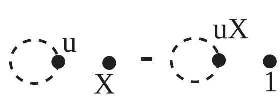
It can be convenient to actually know a set of generators of the cohomology groups of a graph rather than knowing only the isomorphism types of the cohomology groups, for instance when we want to use the exact sequence. The next proposition explains how to produce those in this specific case. It is a corollary of the previous result, we just need to write explicitly what p and f are, namely, $p = (-1)^i\gamma^*$ and $f(x \otimes X) = x \otimes X - (-1)^i\gamma^*(x \otimes X) \otimes 1$.

Corollary 30. *Let G be a graph with a pendant edge. G can be represented by* *, where the S stands for "something". If we know a set of generators of $H^i($$)$, applying the following isomorphisms to a set of generators of $H^i($$)$ produces a set of generators of $H^i($$)$.*

$H^i\Big($$\Big) \xrightarrow{f} \ker(\gamma^*) \xrightarrow{(\bar{\beta}^*)^{-1}} H^i\Big($$\Big)$

$x \mapsto \underbrace{x \otimes X - (-1)^i\gamma^*(x \otimes X) \otimes 1}_{y} \mapsto y$ *seen in* $H^i\Big($$\Big)$

In the above definition of y, the right-most X in $x \otimes X$ means that the isolated vertex v is assigned the value X (similar interpretation for the other term).
The isomorphism can be visualized the following way. x is a sum of terms of the form 

. This picture means that the component that includes this vertex has been assigned the value $u \in \mathcal{M}$. Such a term becomes 

under the action of f, where the element of $\mathcal{M}$ written close to a vertex indicates the label that has been assigned to this component.

We have a similar algorithm when we extend our construction to a larger class of algebras, as explained in remark (48).

The method is illustrated in the following example:

In the array that keeps track of the cohomology groups, see Figure (2.1), the numbers without brackets indicate the number of copies of $\mathbb{Z}$ while the numbers with brackets indicate the number of copies of $\mathbb{Z}_2$.

For instance, in the case of the triangle K_3 illustrated in Figure (2.1), the first column means $H^0(K_3) \cong \mathbb{Z}\{3\}$ ($= H^{0,3}(K_3)$, elements of degree 3). The second column means $H^{1,1}(K_3) \cong \mathbb{Z}\{1\}$ (elements of degree 1 in H^1) and $H^{1,2}(K_3) \cong \mathbb{Z}_2\{2\}$ (elements of degree 2 in H^1) i.e. $H^1(K_3) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}_2\{2\}$.

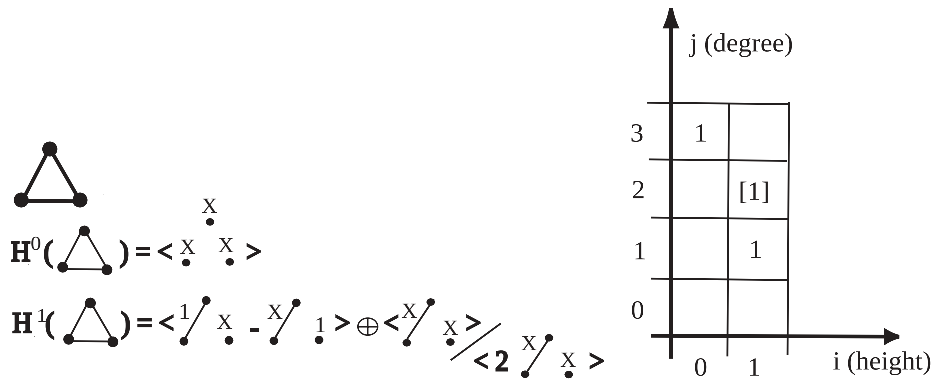


Figure 2.1: K_3 *summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

We can use the method described above to produce the generators of the cohomology groups of a triangle to which a pendant edge has been added and the results are shown in Figure (2.2).

2.6 Trees, circuits graphs

We describe the cohomology groups for several classes of graphs.

Example 31. *Let N_1 be the graph with 1 vertex and no edge. We have $P_{N_1} = \lambda = 1 + q$. The only enhanced states of N_1 are $(\emptyset, 1)$ and $(\emptyset, X)$, which generate $C^0(N_1)$. It follows that $H^0(N_1) \cong \mathbb{Z} \oplus \mathbb{Z}\{1\}$, and $H^i(N_1) = 0$ for all $i \neq 0$.*

Example 32. *More generally, the graph with p vertices and no edges is called the null graph of order p and denoted by N_p. A similar argument implies*

$$H^i(N_p) \cong \begin{cases} [\mathbb{Z} \oplus \mathbb{Z}\{1\}]^{\otimes p} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

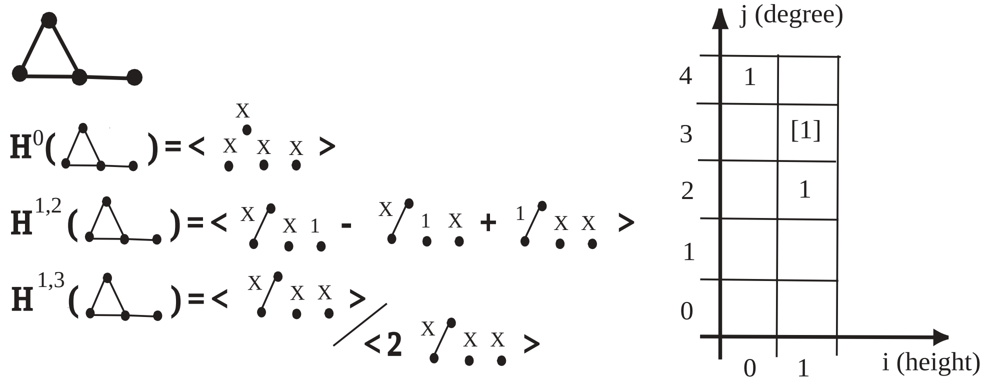


Figure 2.2: *K_3 with a pendant edge added, summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

This also follows from the Künneth type formula in Theorem (25).

Example 33. *Let $G = T_n$, a tree with n edges. We can obtain G by starting from a one point graph, and then adding pendant edges successively. Thus Theorem (28) and Example (31) imply*

$$H^0(T_n) \cong [\mathbb{Z} \oplus \mathbb{Z}\{1\}]\{n\} \cong \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n+1\},\ H^i(T_n) = 0 \textit{ for } i \neq 0.$$

A basis for $H^0(T_n)$ can be described as follows. Let $V(T_n) = \{v_0, v_1, \cdots, v_n\}$ be the set of vertices of T_n. Let $\sigma : V(T_n) \to \{\pm 1\}$ be an assignment of ± 1 to the vertices of T_n such that vertices that are adjacent in G always have opposite signs. It is easy to see that such a σ exists (e.g. let $\sigma(v) = (-1)^{d(v_0,v)}$ where $d(v_0, v)$ is the number of edges in T_n that connect v_0 to v). Furthermore, σ is unique up to multiplication by -1. For each $k = 0, 1, \cdots, n$, let $S_k = (\emptyset, c_k)$ be the enhanced state in which $s = \emptyset$ and c_i assigns 1 to v_k and X to v_j for each $j \neq k$. Let $\varepsilon_1 = \Sigma_{k=0}^{n} \sigma(v_k) S_k \in C^0(T_n)$, and let $\varepsilon_2 = (\emptyset, c)$ be the enhanced state with $s = \emptyset$ and c assigns X to each vertex v_j for $j = 0, \cdots, n$. Then ε_1 is a generator for $\mathbb{Z}\{n\}$ and ε_2 is a generator for $\mathbb{Z}\{n+1\}$ in $H^0(T_n)$. An example is shown below.

Example 34. *Circuit graph with n edges*

Let $G = C_n$, the circuit graph with n edges, also known as the cycle graph.

▲ If $n = 1$, C_1 is the graph with one vertex and one loop. By Proposition (23), $H^i(C_1) = 0$ for each i.

▲ If $n = 2$, C_2 is the graph with two vertices connected by two parallel edges. By Proposition (24), $H^i(C_2) \cong H^i(T_1) \cong \begin{cases} (\mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\}) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$

▲ Next, let us assume $n > 2$.

(In fact, the following method also holds for $n = 2$ but the above method is a much easier way to get the result).

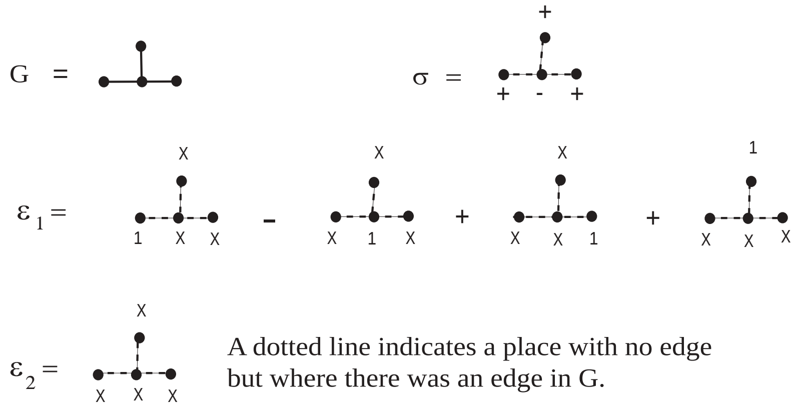


Figure 2.3: *An example of basis for trees*

$\triangle$ We label the vertices of C_n by $v_1, \cdots, v_n$ monotonically so that each v_k is adjacent to v_{k+1} (here $v_{n+1} = v_1$). Let e be the edge $v_1 v_n$. Then $G - e$ is the tree with n vertices $v_1, \cdots, v_n$ connected by a line segment running from v_1 to v_n, and G/e is the circuit graph C_{n-1} with vertices $v_1, \cdots, v_{n-1}$. The exact sequence on (G, e) gives

$$\cdots \to H^{i-1}(G) \to H^{i-1}(G-e) \to H^{i-1}(G/e) \to H^i(G) \to H^i(G-e) \to \cdots$$

For $i \geqslant 2, H^{i-1}(G-e) = H^i(G-e) = 0$ by Example (33). Thus $H^i(G) \cong H^{i-1}(G/e)$, i.e. $H^i(C_n) \cong H^{i-1}(C_{n-1})$ provided if $n \geqslant 2$ and $i \geqslant 2$. Applying this equation repeatedly, we have

$$H^i(C_n) \cong \begin{cases} H^1(C_{n-i+1}) & \text{if } i \leqslant n \\ H^{i-n+1}(C_1) = 0 & \text{if } i \geqslant n. \end{cases}$$

Thus it suffices to determine $H^1(C_n)$ and $H^0(C_n)$ for all n. Again, we examine part of the long exact sequence:

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \to 0$$

Here, the last group $H^1(G-e)$ is 0 because $G-e$ is a tree. This exact sequence implies that

$$H^0(G) \cong \ker \gamma^*, H^1(G) \cong H^0(G/e)/\ker \alpha^* = H^0(G/e)/\mathrm{Im}\gamma^*$$

$\triangle$ Thus we need to understand the map $\gamma^* : H^0(G-e) \to H^0(G/e)$.

This map can be described as follows. An x in $H^0(G-e)$ is the equivalence class of a sum of terms of the form $(\emptyset, c)$ in $C^0(G-e)$. Each of these enhanced state $S = (\emptyset, c)$ is just a coloring of $v_1, \cdots, v_n$ by 1 or X. Under the map γ^*, $S = (\emptyset, c)$ is changed to $(\emptyset, \gamma(c))$ where $\gamma(c)$ is the coloring on $V(G/e)$ defined by $\gamma(c)(v_k) = c(v_k)$ for each $k \neq 1, n$, and

$\gamma(c)(v_1) = c(v_1) * c(v_n)$. Basically, for each each (s, c), γ^* adds the edge e, adjusts the colorings, then contracts e to a point and multiplies the result by $(-1)^i$.

By Example (33), $H^0(G-e) \cong \mathbb{Z}\{n-1\} \oplus \mathbb{Z}\{n\}$ where $\mathbb{Z}\{n-1\}$ is generated by ε_1 and $\mathbb{Z}\{n\}$ is generated by ε_2. It is easy to see that $\gamma^*(\varepsilon_2) = 0$ since all vertices are colored by X in ε_2. As for $\gamma^*(\varepsilon_1)$, it will depend on the parity of n. We have $\varepsilon_1 = S_1 - S_2 + \cdots + (-1)^{n-1} S_n$. For each $k \neq 1, n$, $\gamma^*(S_k) = 0$ since both v_1 and v_n are labeled by X under S_k. For $k = 1$ and $k = n$, $\gamma^*(S_1) = \gamma^*(S_n) = \varepsilon_2'$ where ε_2' is the state of $C^0(G/e)$ that labels every vertex by X. Thus $\gamma^*(\varepsilon_1) = 0$ if n is even, and $\gamma(\varepsilon_2) = 2\varepsilon_2'$ if n is odd.

It follows that ker $\gamma^* = \text{Span}\{\varepsilon_1, \varepsilon_2\}$ if n is even, and ker $\gamma^* = \text{Span}\{\varepsilon_2\}$ if n is odd. Therefore

$$H^0(C_n) \cong \begin{cases} \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n-1\} & \text{if } n \text{ is even and } n \geqslant 2 \\ \mathbb{Z}\{n\} & \text{if } n \text{ is odd and } n > 2. \end{cases}$$

$\triangle$ Next, we determine $H^1(C_n)$ using the same exact sequence. We follow the discussion above. If n is even, $\gamma^* = 0$, and therefore $H^1(G) \cong H^0(G/e) \cong \mathbb{Z}\{n-1\}$. If n is odd, $\text{Im}\gamma^* = 2\mathbb{Z}\{n-1\}$ in $H^0(G/e)$. Therefore $H^1(C_n) \cong H^0(G/e)/\ \text{Im}\gamma^* \cong \mathbb{Z}\{n-1\} \oplus \mathbb{Z}\{n-2\}/2\mathbb{Z}\{n-1\} \cong \mathbb{Z}\{n-2\} \oplus \mathbb{Z}_2\{n-1\}$.

As a summary, we have

$$\text{For } i > 0, H^i(C_n) \cong \begin{cases} \mathbb{Z}_2\{n-i\} \oplus \mathbb{Z}\{n-i-1\} & \text{if } n-i \geqslant 2 \text{ and is even} \\ \mathbb{Z}\{n-i\} & \text{if } n-i \geqslant 2 \text{ and is odd} \\ 0 & \text{if } n-i \leqslant 1. \end{cases}$$

$$\text{For } i = 0, H^0(C_n) \cong \begin{cases} \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n-1\} & \text{if } n \text{ is even and } n \geqslant 2 \\ \mathbb{Z}\{n\} & \text{if } n \text{ is odd and } n \geqslant 2 \\ 0 & \text{if } n = 1. \end{cases}$$

Computational results 35. *The following table illustrate our computational result (up to $n = 6$ and $i = 4$) for circuit graphs. We denote P_n the circuit graph on n vertices where P stands for polygon because the notation C_n has already been used for the chain groups.*

$n\backslash i$	H^0	H^1	H^2	H^3	H^4
P_1	*0*	*0*	*0*	*0*	*0*
P_2	$\mathbb{Z}\{2\} \oplus \mathbb{Z}\{1\}$	*0*	*0*	*0*	*0*
P_3	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	*0*	*0*	*0*
P_4	$\mathbb{Z}\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	*0*	*0*
P_5	$\mathbb{Z}\{5\}$	$\mathbb{Z}_2\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	*0*
P_6	$\mathbb{Z}\{6\} \oplus \mathbb{Z}\{5\}$	$\mathbb{Z}\{5\}$	$\mathbb{Z}_2\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$

We note that, for all $n \geqslant 3$, $H^*(C_n)$ contains torsion. We will analyze such phenomenon for a general graph later on.

2.7 Vanishing theorem

Theorems (36) and (37) indicate which cohomology groups may be non-trivial.

Theorem 36. *Let G be a graph with p vertices, $p \geq 2$.*
If $i > p-2$, then $H^i(G) = 0$.

This result was first stated by Michael and Sergei Chmutov.

Note that we have to exclude the case $p = 1$ (or the case $i = 0$) otherwise the graph N_1 made of one isolated vertex would give us a contradiction since $H^0(N_1) \neq 0$ but $i = 0 > 1-2$.

Proof. Let n be the number of edges of G. We are going to prove this result by induction on n.

▲ If $n = 0$, G is the null graph on p vertices. We have already shown that $H^i(N_p) = 0$ unless $i = 0$, so the induction hypothesis is satisfied.

▲ Let $n \geqslant 1$. Assume the induction hypothesis holds for $n-1$.
△ *Case 1:* $p > 2$
Assume $i > p-2$. We need to show that $H^i(G) = 0$.
The exact sequence on (G, e) gives

$$\cdots \to H^{i-1}(G/e) \to H^i(G) \to H^i(G-e) \to \cdots$$

Since $G-e$ has one less edge than G and p vertices with $p \geq 2$, the induction hypothesis applies and $H^i(G-e) = 0$.

Also, $i > p-2$ implies $i-1 > (p-1)-2$. Since G/e has one less edge than G and $p-1$ vertices with $p-1 \geq 2$, we can apply the induction hypothesis and we get $H^{i-1}(G/e) = 0$. Substituting these results in the exact sequence proves that $H^i(G) = 0$.

△ *Case 2:* $p = 2$
Either G has a loop or G has no loop. If G has a loop then all the cohomology groups are trivial so the induction hypothesis is satisfied. If G has no loop, G is a graph with two vertices connected by n edges. Since we can delete multiple edges without changing the cohomology groups, G has the same cohomology groups as a tree with one edge so all the cohomology groups are trivial except $H^0(G)$ so, again, the induction hypothesis is satisfied. □

2.8 Thickness of the cohomology

Theorem 37. *Let G be a graph with p vertices and μ components.*
Then $H^{i,j}(G) = 0$ unless $p - \mu \leqslant i + j \leqslant p$.

This extends the corresponding result for connected graphs that was first stated by Michael and Sergei Chmutov.

Before proving the theorem, we would like to illustrate its meaning by two examples. In short, the theorem says that the cohomology is concentrated along $\mu + 1$ diagonals in the case of a graph with μ components. The "$\mu + 1$ diagonals" language refers to the way of keeping track of the cohomology groups illustrated in Figure (2.4). In the array that keeps track of the cohomology groups, the numbers without brackets indicate the number of copies of $\mathbb{Z}$ while the numbers with brackets indicate the number of copies of $\mathbb{Z}_2$. For instance, in the case of the cyclic graph P_6 (see figure (2.4) below), the [1] in position $i = 2, j = 4$ means $H^{2,4}(P_6) \cong \mathbb{Z}_2\{4\}$ and the 1 in position $i = 2, j =$ means $H^{2,3}(P_6) \cong \mathbb{Z}\{3\}$, so $H^2(P_6) \cong \mathbb{Z}\{3\} \oplus \mathbb{Z}_2\{4\}$.

For a connected graph, the theorem says that the cohomology is always concentrated along two diagonals, which are $i + j = p = 6$ and $i + j = p - 1 = 5$ in the particular case of the graph P_6.

This example also illustrates that all the non trivial cohomology groups have height i such that $0 \leqslant i \leqslant 6 - 2$ as predicted by theorem (36).

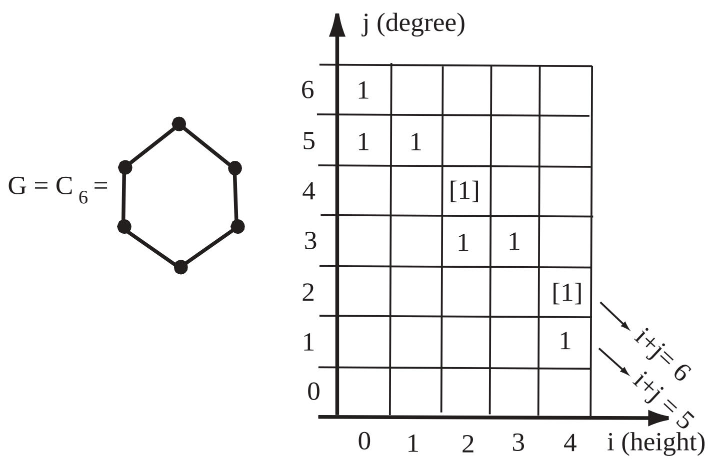


Figure 2.4: P_6 *summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

The case of the graph which is the disjoint union of two triangles and an isolated vertex is illustrated in Figure (2.5) below. This graph has three components and the cohomology groups are organized along four diagonals.

Proof. Let n be the number of edges of G. We are going to prove this result by induction on n.

▲ If $n = 0$, G is the null graph on p vertices. We have already shown that $H^0(N_p) = [\mathbb{Z} \oplus \mathbb{Z}\{1\}]^{\otimes p}$. Expanding this tensor product will yield groups with degrees ranging from

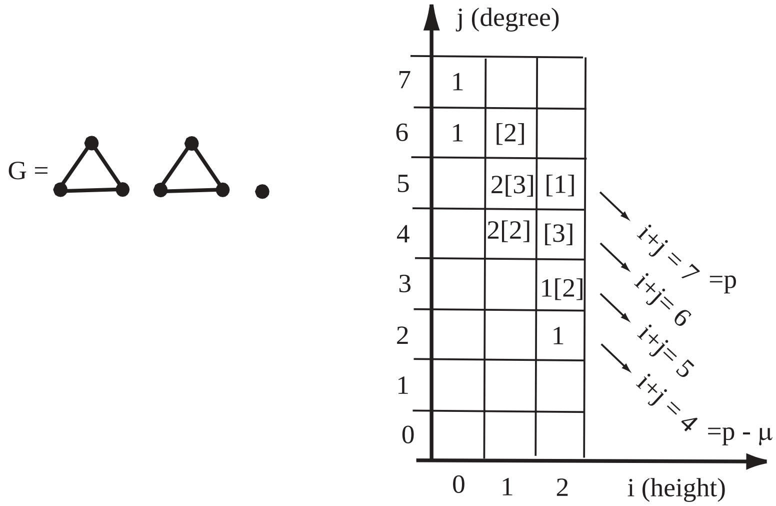


Figure 2.5: *Two triangles and an isolated vertex, summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

0 to p. In other words, $p - \mu = 0 \leqslant 0 + j \leqslant p$. All the other cohomology groups are trivial, so the induction hypothesis is satisfied.

▲ Assume the induction hypothesis holds for all $n' < n$.

If G has a loop, the induction hypothesis is satisfied, so we can assume from now on that G has no loop.

If G has multiple edges, the graph G' obtained from G by replacing multiple edges by single edges has fewer edges so we can use the induction hypothesis for G'. Since G and G' have the same cohomology groups, the same number of components and the same number of vertices, this proves the result for G. From now on in this proof, we assume that G has no loop and no multiple edges.

Case 1: G is a forest

Assume G is a forest made of μ trees for a total of n edges. We denote such a forest $F_{\mu,n}$. It is clear that $F_{\mu,n}$ has $\mu + n$ vertices.

The cohomology groups of $F_{\mu,n}$ are obtained from the cohomology groups of N_μ, the graph with μ vertices and no edges, by adding n pendant edges. Therefore, the cohomology groups are trivial except for $H^0(F_{\mu,n}) \cong [\mathbb{Z} \oplus \mathbb{Z}\{1\}]^{\otimes \mu}\{n\}$. Expanding the tensor product, we get groups of the form $\mathbb{Z}\{j\}$ with $0 \leqslant j \leqslant \mu$. If we now increase the degrees by n, we get groups of the form $\mathbb{Z}\{j\}$ with $n \leqslant j \leqslant \mu + n$. Substituting $n = p - \mu$ in the above yields $p - \mu \leqslant 0 + j \leqslant p$, so the assumption hypothesis is satisfied (without using induction).

Case 2: G is a not forest

Since we assumed that G has no loop and no multiple edges, G is a not forest means it contains a cycle of order $\geqslant 3$. Let e be an edge on this cycle. The edge e is not an isthmus so $G-e$ also has μ components. Of course, G/e has μ components no matter whether e is not an isthmus or not.

The exact sequence on (G,e) gives

$$\cdots \to H^{i-1,j}(G) \to H^{i-1,j}(G-e) \to H^{i-1,j}(G/e) \to H^{i,j}(G) \to H^{i,j}(G-e) \to \cdots$$

Note that we don't have to worry about having $i-1 \geq 0$ since we allow negative heights for cohomology groups as explained in Remark (22).

In the exact sequence all the graph have μ components as observed earlier. Also note that since $G-e$ and G/e have one less edge than G, the induction hypothesis applies to them. For $G-e$, the induction hypothesis says that $H^{i,j}(G-e)=0$ unless $p-\mu \leqslant i+j \leqslant p$. For G/e, the induction hypothesis says that $H^{i,j}(G/e)=0$ unless $p-1-\mu \leqslant i+j \leqslant p-1$. We use these facts to show that $H^{i,j}(G)=0$ whenever $i+j>p$ or $i+j<p-\mu$.

▲ Assume that $i+j>p$. We need to show that $H^{i,j}(G)=0$.

The exact sequence on (G,e) gives

$$\cdots \to H^{i-1,j}(G/e) \to H^{i,j}(G) \to H^{i,j}(G-e) \to \cdots$$

$i+j>p$ so $H^{i,j}(G-e)=0$. Also $i+j>p$ implies $i-1+j>p-1$ so $H^{i-1,j}(G/e)=0$. Therefore the sequence $0 \to H^{i,j}(G) \to 0$ is exact so $H^{i,j}(G)=0$.

▲ Assume that $i+j<p-\mu$. We need to show that $H^{i,j}(G)=0$

The exact sequence on (G,e) gives

$$\cdots \to H^{i-1,j}(G/e) \to H^{k,j}(G) \to H^{i,j}(G-e) \to \cdots$$

$p-\mu<i+j$ so $H^{i,j}(G-e)=0$. Also $i+j<p-\mu$ implies $(i-1)+j<(p-1)-\mu$ so $H^{i-1,j}(G/e)=0$. Therefore the sequence $0 \to H^{i,j}(G) \to 0$ is exact so $H^{i,j}(G)=0$.

□

2.9 0-cohomology theorem

Theorem 38. *If G is a loopless connected graph with p vertices, then*

$$H^0(G) \cong \begin{cases} \mathbb{Z}\{p\} \oplus \mathbb{Z}\{p-1\} & \textit{for bipartite graphs,} \\ \mathbb{Z}\{p\} & \textit{for non-bipartite graphs} \end{cases}$$

This result was discovered independently by Michael and Sergei Chmutov.

Proof. Let G be a loopless connected graph with p vertices. Since G is connected, by Theorem (37), we know that among the $H^{0,j}(G)$ groups, the only ones that might not be trivial are the ones corresponding to degrees $p-1$ and p.

▲ **degree** p

A basis element for $C^{0,p}(G)$ has p components and degree p. So there is only one basis element, the one for which all the vertices are assigned the value X. Let's denote it b. Since there are no loops, adding an edge will connect two vertices with X and therefore the image of b under the differential is zero. Hence $H^{0,p}(G) =< b >\cong \mathbb{Z}\{p\}$

▲ **degree** $p-1$

△ The basis elements of $C^{0,p-1}(G)$ have p components and degree $p-1$ which means that all vertices are assigned the value X except one, say v, which is assigned the value 1. We denote such a basis element by b_v.

We also need to describe the basis elements of the target space $C^{1,p-1}(G)$. First, note that they have $p-1$ components because, since there are no loops, adding one edge automatically decreases the number of components by one. Therefore for degree reasons, all the components are assigned the value X. The basis element for which the present edge is e and with X assigned to all the components is denoted a_e. We have

$$d(b_v) = \sum_{e=vw\in E(G)} a_e$$

because adding an edge that doesn't have v as one of its vertices will connect two vertices with X and therefore yield zero.

△ **Claim**: Let $x = \sum_{v\in V(G)} \lambda_v b_v \in \ker d^0 = H^0(G)$. If some vertices v and w (not necessarily distinct) are connected by a walk of ℓ edges, then the corresponding coefficients in x satisfy $\lambda_v = (-1)^\ell \lambda_w$.

Proof of Claim. The proof is by induction on ℓ.

⋄ $\ell = 1$ means that we are talking about adjacent vertices.

$d(x) = 0 = \sum_{v\in V(G)} \lambda_v \sum_{e=vw\in E(G)} a_e = \sum_{e=vw\in E(G)} (\lambda_v + \lambda_w) a_e$ because by assumption, there are no loops so each edge $e = vw$ has two distinct endpoints v and w. Thus, $\lambda_v = -\lambda_w$ for all adjacent vertices $v, w \in V(G)$.

⋄ Assume the result holds for ℓ. Let v and w be vertices that are connected by a walk of $\ell+1$ edges. Denote by u the one before last vertex in this walk from v to w. The induced walk from v to u has length ℓ so by induction hypothesis, $\lambda_v = (-1)^\ell \lambda_u$. Using the first step of the induction, we get $\lambda_u = -\lambda_w$. These two results together show that $\lambda_v = (-1)^{\ell+1}\lambda_w$.

△ **Case 1: G is bipartite**:

⋄ Since G is bipartite, its vertices can be partitioned into two sets Y and Z such that each edge consists of one vertex from each set. Let $b = \sum_{v\in Y} b_v - \sum_{w\in Z} b_w$. Applying the per-edge map that adds the edge $e = vw$ where $v \in Y$ and $w \in Z$ will yield $a_e - a_e = 0$. This holds for all per-edge map so $d^{0,p-1}(b) = 0$ and $< b > \subseteq H^{0,p-1}(G)$.

⋄ Let $x = \sum \lambda_u b_u \in \ker d^{0,p-1} = H^{0,p-1}(G)$. Without loss of generality, we can assume that at least one of the vertices is in Y. Denote v such a vertex. For any vertex w, since G is bipartite and connected, if w is in Y, there is an even v, w path so by the claim, $\lambda_v = \lambda_w$. By the claim again, if w is in Z, there is an odd v, w path so by the claim, $\lambda_v = -\lambda_w$. Hence, $x = \lambda_v b$ so $H^{0,p-1}(G) \subseteq < b >$.

⋄ Combining these results yields $H^{0,p-1}(G) =< b > \cong \mathbb{Z}\{p-1\}$.

△ **Case 2: G is not bipartite**:

Let $x = \sum \lambda_v b_v \in \ker d^{0,p-1} = H^{0,p-1}(G)$. A well-known result of graph theory says that a non-bipartite graph has an odd cycle. Pick one vertex, say v_0, on this cycle. There is a path from v_0 to itsself with an odd number of edges. By the claim, it means that $\lambda_{v_0} = -\lambda_{v_0}$ so $\lambda_{v_0} = 0$. The claim also implies that all the λ_v's are equal or opposite to λ_{v_0} so all are zero. Hence, $x = 0$ and $H^{0,p-1}(G) = kerd^{0,p-1} = 0$

□

Chapter 3

Extension of this construction to other algebras

Khovanov constructed a graded cohomology theory for classical links and showed it yields the Jones polynomial by taking the graded Euler characteristic. In Chapter (1) we constructed a graded cohomology theory for graphs and showed it yields the Chromatic polynomial by taking the graded Euler characteristic. Both constructions depend on a given graded algebra (or bi-algebra for links) which is the building block for the cochain groups in the cochain complex. However, the amounts of choices of algebras are quite different. In the case of links, the choices are quite limited due to the requirement of invariance under Redemeister moves. In the case of graphs, the choices are abundant. The algebra used in our cohomology for graphs, $\mathbb{Z}[X]/(X^2)$, is the simplest natural choice. Note that we started our construction with $\mathbb{Z}[X]/(X^2)$ seen as a $\mathbb{Z}$-module but then we equipped it with a multiplication m when we defined the differential that turned it into a commutative $\mathbb{Z}$-algebra. The purpose of this chapter is to show that our construction can be extended to a large class of algebras.

In section (3.1), we explain the definition of the cochain complex, and show that the Euler characteristic of the cohomology groups is equal to the chromatic polynomial of the graph evaluated at $\lambda = q\dim\mathcal{A}$ where $q\dim\mathcal{A}$ is the graded dimension of the algebra $\mathcal{A}$. In section (3.1), we discuss some basic properties of our cohomology groups. In particular, we construct a long exact sequence which can be considered as a categorification of the deletion-contraction rule of the chromatic polynomial. In section (3.1), we show some computational examples. In particular, we note the existence of an order 3 torsion in our cohomology groups, when the algebra is $\mathbb{Z}[X]/(X^3)$. This is in stark contrast with the cohomology groups in our cohomology for graphs when $\mathbb{Z}[X]/(X^2)$ and in Khovanov cohomology for links, where the computations suggest no odd torsion can occur [S04].

The results of this section are covered in [HR05]. We wish to thank Mikhail Khovanov for his suggestions and comments.

3.1 The Construction

Definitions: Graded algebra, graded dimension

All the algebras $\mathcal{A}$ that we will consider here will be algebras over $\mathbb{Z}$[1] with a unit $1_{\mathcal{A}}$.

Definition 39. *A* graded $\mathbb{Z}$-algebra $\mathcal{A}$ *is a $\mathbb{Z}$-algebra with direct sum decomposition $\mathcal{A} = \oplus_{j=0}^{\infty} A_j$ into $\mathbb{Z}$-submodules such that $a_i a_j \in A_{i+j}$ for all $a_i \in A_i$ and $a_j \in A_j$. The elements of A_j are called* homogeneous *elements of degree j.*

From now on, all algebras $\mathcal{A}$ will satisfy the following conditions :

Assumptions 40. $\mathcal{A} = \oplus_{j=0}^{\infty} A_j$ *is a commutative graded algebra over $\mathbb{Z}$ such that each A_j, the set of degree j homogeneous elements, is a free $\mathbb{Z}$-module of finite rank. In particular, $\mathcal{A}$ is a graded $\mathbb{Z}$-module whose graded dimension is the power series $q\dim \mathcal{A} = \sum_j q^j \, rank A_j$.*

Note that these assumptions can sometimes be relaxed. For instance, the construction can still be made even if there is no identity or if the A_i's are not free.

The construction

Let $\mathcal{A}$ be an algebra satisfying Assumptions (40). The construction is similar to the one in Chapter (1), the difference being that the algebra $\mathbb{Z}[X]/(X^2)$ is replaced by $\mathcal{A}$.

Let G be a graph and $E = E(G)$ be the edge set of G. Let $n = |E|$ be the cardinality of E. We fix an ordering on E and denote the edges by $e_1, \cdots, e_n$. Consider the n-dimensional cube $\{0,1\}^E = \{0,1\}^n$. Each vertex α of this cube corresponds to a subset $s = s_\alpha$ of E, where $e_i \in s_\alpha$ if and only if $\alpha_i = 1$. The height $|\alpha|$ of α, is defined by $|\alpha| = \sum \alpha_i$, which is also equal to the number of edges in s_α.

The cochain groups

For each vertex α of the cube, we associate the graded $\mathbb{Z}$-module $C_\alpha(G)$ as follows. Consider $[G : s]$, the graph with vertex set $V(G)$ and edge set s. We assign a copy of $\mathcal{A}$ to each component of $[G : s]$ and then take tensor product over the components. Let $C_\alpha(G)$ be the resulting graded $\mathbb{Z}$-module, with the induced grading from $\mathcal{A}$. Therefore, $C_\alpha(G) \cong \mathcal{A}^{\otimes k}$ where k is the number of components of $[G : s]$. Similarly to our previous construction of the cochain groups when $\mathcal{A} = \mathbb{Z}1 \oplus \mathbb{Z}X$, we define the i^{th} chain $\mathbb{Z}$-module to be

$$C^i(G) := \oplus_{|\alpha|=i} C_\alpha(G).$$

[1]Of course, algebras over $\mathbb{Z}$ are nothing but rings but we prefer to think of them as $\mathbb{Z}$-algebras rather than rings because most results can be extended to $\mathcal{R}$-algebras, for a commutative ring $\mathcal{R}$ with 1. Similarly, we talk about $\mathbb{Z}$-modules rather than abelian groups because this setting allows easier generalization to $\mathcal{R}$-modules. More about this generalization from $\mathbb{Z}$ to $\mathcal{R}$ will be in [HR05].

Keep in mind that $C^i(G)$ depends on the algebra $\mathcal{A}$. Thus one may want to denote it by $C^i_{\mathcal{A}}(G)$. However, we will omit the letter $\mathcal{A}$ unless there is an ambiguity. Also, we sometimes interchange the notions α and s. Thus $C_\alpha(G)$ is sometimes denoted by $C_s(G)$. This certainly should not cause any confusion.

The differential

To define the differential maps d^i, we need to make use of the edges of the cube $\{0,1\}^E$. Each edge ξ of $\{0,1\}^E$ can be labeled by a sequence in $\{0,1,*\}^E$ with exactly one $*$. The tail of the edge is obtained by setting $*=0$ and the head is obtained by setting $*=1$. The height $|\xi|$ is defined to be the height of its tail, which is also equal to the number of 1's in ξ.

Given an edge ξ of the cube, let α_1 be its tail and α_2 be its head. The per-edge map $d_\xi : C_{\alpha_1}(G) \to C_{\alpha_2}(G)$ is the $\mathbb{Z}$-linear map defined as follows. For $j = 1$ and 2, the $\mathbb{Z}$-module $C_{\alpha_j}(G) = \mathcal{A}^{\otimes k_j}$ where k_j is the number of connected components of $[G : s_j]$ (here $s_j = s_{\alpha_j}$). Let e be the edge with $s_2 = s_1 \cup \{e\}$.

$\diamond$ If e joins a component of $[G : s_1]$ to itself, then $k_1 = k_2$ and the components of $[G : s_1]$ and the components of $[G : s_2]$ naturally correspond to each other. We let d_ξ to be the identity map.

$\diamond$ If e joins two different components of $[G : s_1]$, say E_1 to E_2 where $E_1, E_2, \cdots, E_{k_1}$ are the components of $[G : s_1]$, then $k_2 = k_1 - 1$ and the components of $[G : s_2]$ are $E_1 \cup E_2 \cup \{e\}, E_3, \cdots, E_{k_1}$. We define d_ξ to be the identity map on the tensor factors coming from $E_3, \cdots, E_{k_1}$, and d_ξ on the remaining tensor factors to be the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ sending $x \otimes y$ to xy.

Now, we define the differential $d^i : C^i(G) \to C^{i+1}(G)$ by $d^i = \sum_{|\xi|=r} (-1)^\xi d_\xi$, where $(-1)^\xi = (-1)^{\sum_{i<i_0} \xi_i}$ where i_0 is the position of the star in ξ, i.e. $\sum_{i<i_0} \xi_i$ is the number of 1's before $*$ in ξ.

Theorem 41. *Let G be a finite graph, and let $\mathcal{A}$ be an algebra satisfying Assumptions (40). Then*

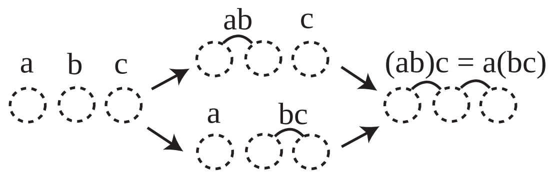
(a) $0 \to C^0(G) \xrightarrow{d^0} C^1(G) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(G) \to 0$ *is a graded cochain complex whose differential is degree preserving.*

(b) The cohomology groups $H^i(G) (= H^i_{\mathcal{A}}(G))$ are independent of the ordering of the edges of G, and therefore are invariants of the graph G.

(c) The graded Euler characteristic of the cochain complex is equal to the chromatic polynomial of the graph G evaluated at $\lambda = q \dim \mathcal{A}$, i.e. $\chi_q(C) = \sum\limits_{0 \le i \le n} (-1)^i q\dim(H^i) = \sum\limits_{0 \le i \le n} (-1)^i q\dim(C^i) = P_G(q \dim \mathcal{A})$

Proof. The proof is rather standard and similar to the one in the previous case. We sketch the ideas here.

(a). To prove this defines a cochain complex, we need to show that d is a differential. That is, $d \circ d = 0$. This is done in two steps. First, we verify that the maps d_ξ makes the cube commutative, a fact follows from the associativity of the algebra, as illustrated below.



where a dotted circle represents a connected component of the graph.

Second, the signs $(-1)^\xi$ in d allow us to cancel out all terms in $d \circ d$. Thus $d \circ d = 0$.

Note that here is no canonical order on the components of $[G : s]$ so we need the multiplication to be commutative.

To show that d is degree preserving, we note that the multiplication map on $\mathcal{A}$ is always degree preserving, which then implies each map d_ξ is degree preserving, and therefore so is d.

(b). This is similar to Theorem (17). Each permutation of the edges of G is a product of transpositions of the form $(k, k+1)$. An explicit isomorphism can be constructed for each such transposition. In fact, this shows that the isomorphism class of the cochain complex is an invariant of the graph.

(c). First, we already showed in the proof of Proposition (10) that $\sum\limits_{0 \le i \le n} (-1)^i \cdot q\dim(H^i) = \sum\limits_{0 \le i \le n} (-1)^i \cdot q\dim(C^i)$. Next, we note that $q\dim C_\alpha(G) = q\dim \mathcal{A}^{\otimes k} = (q\dim \mathcal{A})^k$ where k is the number of connected components of $[G : s]$. Taking direct sum over $s \subseteq E(G), |s| = i$, and then taking alternating sum over i, we obtain the equation $\sum\limits_{0 \le i \le n} (-1)^i \cdot q\dim(C^i) = P_G(q\dim \mathcal{A})$ by the state sum (1.2).

□

Remark 42. *(a) The above graded cochain complex can easily be seen to be a bi-graded cochain complex. Let $\mathcal{C}^{i,j}(D)$ be the submodule of $\mathcal{C}^i(D)$ consisting of homogeneous elements with degree j. Let $\mathrm{d}^{i,j}$ be the restriction of d^i to elements with degree j. For each j we have a cochain complex*

$$0 \to \mathcal{C}^{0,j}(D) \xrightarrow{d^{0,j}} \mathcal{C}^{1,j}(D) \xrightarrow{d^{1,j}} \cdots \xrightarrow{d^{n-1,j}} \mathcal{C}^{n,j}(D) \to 0$$

The direct sum of these cochain complexes, with the obvious gradings, is equal to the cochain complex in Theorem (41). The different gradings don't interfere hence $C^i(G) = \oplus_j C^{i,j}(G)$ and $H^i(G) = \oplus_j H^{i,j}(G)$.
(b) Our cochain complexes can also be described in terms of enhanced states. One defines an enhanced state S of G to be a pair (s, c) where $s \subseteq E(G)$ and c is an assignment of an element of $\mathcal{A}$ to each connected component of $[G : s]$. One identifies S with the element

$c(E_1) \otimes \cdots \otimes c(E_k)$ of $C^s(G) = \mathcal{A}^{\otimes k}$, where $E_1, \cdots, E_k$ are components of $[G : s]$. Thus $C^i(G)$ is generated by states with $|s| = i$. When each $c(E_i)$ is a homogeneous element of $\mathcal{A}$, we say the coloring c and the enhanced state S are homogeneous, and we define its degree to be $j(S) = \sum_i \deg c(E_i)$. It is easy to see that $C^{i,j}$ above is generated by all homogeneous enhanced states S with $i(S) = i, j(S) = j$. The differential of each enhanced state is then defined to be the operation of adding each edge not in s, adjusting the coloring c using the multiplication on $\mathcal{A}$, and then taking the summation over the edges in $E(G) - s$ with appropriate ± 1 signs in front of each term.

Remark 43. *The definition of the differential can be generalized the following way. Remember that, in the above construction, we set the per-edge map d_ξ to be the identity if adding the edge e doesn't change the number of components. However in this case we can allow d_ξ to be any degree preserving $\mathbb{Z}$-linear map $f : \mathcal{A} \to \mathcal{A}$ such that $f(xy) = f(x)f(y)$, i.e. any degree preserving $\mathbb{Z}$-algebra map. The requirement that the cube commutes before sign assignments forces the condition that f respects multiplication and the requirement that the differential of the resulting cochain complex is degree preserving forces f to have this same property.*
We will use the notation $d = (m, f)$ to indicate that the differential is defined using the multiplication m when adding an edge decreases the number of edges and the map f otherwise. If we don't specify, it means that we are using $d = (m, id)$, as explained above. More on this will be explained in [HR05].

3.2 Some properties

In Chapter (2) we proved various basic properties for the graph cohomology groups when the algebra is $\mathbb{Z}[X]/(X^2)$. These properties can be carried over here for general algebras. The most interesting property is a long exact sequence, which can be considered as a categorification for the deletion-contraction rule for the chromatic polynomial.

The long exact sequence comes from a short exact sequence of graded chain homomorphisms

$$0 \to C^{i-1}(G/e) \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G - e) \to 0$$

which we explain here. Basically α is the map that puts the edge e back, and β is the projection map that kills every state containing e. A more precise description is given below. First, we order the edges of G so that e is the last edge. This induces natural orderings on the edge sets of G/e and $G-e$ by deleting e from the list. For each $s \subseteq E(G/e)$, let $\tilde{s} = s \cup \{e\}$. Then $\tilde{s} \subseteq E(G)$. Recall that $C_s(G/e)$ (resp. $C_{\tilde{s}}(G)$) is the tensor product of $\mathcal{A}$ taken over components of $[G/e : s]$ (resp. $[G : \tilde{s}]$). The components of $[G/e : s]$ and the components of $[G : \tilde{s}]$ are the same except for the one involving e where they are related by a contraction of e. Thus we have $C_s(G/e) \cong C_{\tilde{s}}(G)$ via a natural isomorphism, since the tensor factors naturally correspond to each other. Let $\alpha|_{C_s(G/e)} : C_s(G/e) \to C_{\tilde{s}}(G)$ be this isomorphism. Taking direct sum over s, we obtain the homomorphism $\alpha : C^{i-1}(G/e) \to C^i(G)$.

Next, we explain the map $\beta : C^i(G) \to C^i(G-e)$. We have $C^i(G) = \oplus_{|s|=i} C_s(G)$. If $e \notin s$, s is automatically a subset of $E(G-e)$. We have $C_s(G) = C_s(G-e)$ since the graphs $[G:s]$ and $[G-e:s]$ are identical. The map β acts like the identity map from $C_s(G)$ to $C_s(G-e)$. If $e \in s$, we let $\beta|_{C_s(G)}$ be the zero map. Taking direct sum over s with $|s| = i$, we obtain the map $\beta : C^i(G) \to C^i(G-e)$. A standard diagram chasing argument shows that this defines a short exact sequence of cochain complexes. Thus we have

Theorem 44. *Let G be a graph, and e be an edge of G.*
(a)For each i, there is a short exact sequence of graded chain homomorphisms: $0 \to C^{i-1}(G/e) \xrightarrow{\alpha} C^i(G) \xrightarrow{\beta} C^i(G-e) \to 0$, and therefore by the zig-zag lemma,
(b) it induces a long exact sequence of cohomology groups: $0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} \ldots \to H^i(G) \xrightarrow{\beta^*} H^i(G-e) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} \ldots$

Taking the alternating sum of the graded dimensions in the above long exact sequence, we obtain the deletion-contraction rule. It is in this sense that the long exact sequence is considered as a categorification of the deletion-contraction rule.

It is useful to understand the following geometric description of the maps α^*, β^*, and γ^*: α^* expands the edge e, β^* is the projection map that kills every state containing e, γ^* adds the edge e, contracts it to a point, and then multiply the corresponding term by $(-1)^i$. In all cases, there is a natural coloring that goes with the new state using the multiplication on $\mathcal{A}$.

Other basic properties can follow either from this exact sequence or from the definition. We have

Corollary 45. *(a) If a graph has a loop then all the cohomology groups are trivial.*
(b) The cohomology groups are unchanged if all the multiple edges of a graph are replaced by single edges.

Proof. (a). In the long exact sequence

$$\ldots \to H^{i-1}(G-e) \xrightarrow{\gamma^*} H^{i-1}(G/e) \xrightarrow{\alpha^*} H^i(G) \xrightarrow{\beta^*} H^i(G-e) \xrightarrow{\gamma^*} H^i(G/e) \xrightarrow{\alpha^*} \ldots$$

we have $G/e = G-e$ and the map γ^* is the identity map multiplied by $(-1)^i$. It follows that $H^i(G) = 0$ for each i.

(b) Let e_1 and e_2 be two edges connecting the same pair of vertices in G. In the exact sequence

$$H^{i-1}(G/e_2) \xrightarrow{\alpha^*} H^k(G) \xrightarrow{\beta^*} H^i(G-e_2) \xrightarrow{\gamma^*} H^i(G/e_2)$$

the graph G/e_2 contains a loop coming from e_1. Therefore $H^{i-1}(G/e_2) = H^i(G/e_2) = 0$. It follows that $H^i(G) \cong H^i(G - e_2)$. One can repeat this process until there is no redundant edge in G. $\square$

Next, we consider the effect of adding a pendant edge to a graph. Recall that a *pendant vertex* in a graph is a vertex of degree one, and a *pendant edge* is an edge connecting a pendant vertex to another vertex. Let e be a pendant edge in a graph G, then $P_G(\lambda) = (\lambda - 1)P_{G/e}(\lambda)$. An analogous equation on the cohomology level is given below.

First we prove an algebraic lemma.

Lemma 46. *Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ that satisfies Assumptions (40). Then $\mathcal{A}$ has an identity which generates a direct summand as a $\mathbb{Z}$-module, that is, $\mathcal{A} \cong \mathbb{Z}1_{\mathcal{A}} \oplus \mathcal{A}'$ as a $\mathbb{Z}$-module, where $\mathbb{Z}1_{\mathcal{A}}$ is the $\mathbb{Z}$-module generated by the identity of $\mathcal{A}$.*

Proof. We use the following theorem:

If F is free abelian group of finite rank n and H is a non-zero sub-group of F, then there exists a basis $\{e_1, e_2, \cdots, e_s\}$ of F, an integer r ($1 \leqslant r \leqslant s$) and positive integers $d_1, d_2 \cdots, d_r$ such that $d_1|d_2|\cdots|d_r$ and H is free abelian with basis $\{d_1e_1, d_2e_2, \cdots, d_re_r\}$.

By assumption, $\mathcal{A} = \oplus_{j=0}^{\infty} A_j$ where each A_j is a free $\mathbb{Z}$-module of finite rank. The fact that $1_{\mathcal{A}}^2 = 1_{\mathcal{A}}$ forces the degree of $1_{\mathcal{A}}$ to be zero so $1_{\mathcal{A}} \in A_0$. The previous theorem applied to $F = A_0$ and $H = \mathbb{Z}1_{\mathcal{A}}$ yields a basis $\{e_1, e_2, \cdots, e_s\}$ for A_0 such that $de_1 = 1_{\mathcal{A}}$ for some positive integer d.

It is enough to show that $d = 1$. We have $1_{\mathcal{A}} = (1_{\mathcal{A}})^2$ so $1_{\mathcal{A}} = de_1 = d^2e_1^2$. We write e_1^2 as a linear combination of the basis elements: $e_1^2 = \alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_s e_s$. This implies $1_{\mathcal{A}} = d^2e_1^2 = d^2\alpha_1 e_1 + d^2\alpha_2 e_2 + \cdots d^2\alpha_s e_s$. On the other hand $1_{\mathcal{A}} = de_1$. The uniqueness of coefficients imply the equality $d = d^2\alpha_1$ in $\mathbb{Z}$ so $d = 1$. Therefore $\mathcal{A}' = \widetilde{\mathcal{A}} \oplus [\oplus_{j\geqslant 1} A_j]$ where $\widetilde{\mathcal{A}}$ is the $\mathbb{Z}$-module generated by $\{e_2, e_2, \cdots, e_s\}$ satisfies $\mathcal{A} \cong \mathbb{Z}1_{\mathcal{A}} \oplus \mathcal{A}'$. $\square$

Corollary 47. *Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ that satisfies Assumptions (40). By lemma (46) $\mathcal{A} \cong \mathbb{Z}1_{\mathcal{A}} \oplus \mathcal{A}'$. We have $H^i(G) \cong H^i(G/e) \otimes \mathcal{A}'$ where e is a pendant edge of G.*

Proof. Consider the operations of contracting and deleting e in G. Denote the graph G/e by G_1. We have $G/e = G_1$, and $G - e = G_1 \sqcup \{v\}$, where v is the end point of e with $\deg v = 1$. Consider the exact sequence

$$\cdots H^{i-1}(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^{i-1}(G_1) \xrightarrow{\alpha^*} H^i(G) \xrightarrow{\beta^*} H^k(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to \cdots$$

We need to understand the map

$$H^i(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1)$$

It is easy to compute the cohomology groups for the one point graph $\{v\}$(see the first example in the next section). We have $H^0(\{v\}) \cong \mathcal{A}$ and $H^i(\{v\}) = 0$ for all $i > 0$. Thus, the Künneth type formula below implies

$$H^i(G_1 \sqcup \{v\}) \cong H^i(G_1) \otimes \mathcal{A}$$

by a natural isomorphism h_*. By assumption, $\mathcal{A} \cong \mathbb{Z} \oplus \mathcal{A}'$. We identify $H^i(G_1 \sqcup \{v\})$ with $H^i(G_1) \otimes (\mathbb{Z} \oplus \mathcal{A}')$. The map $\gamma^* : H^i(G_1 \sqcup \{v\}) \to H^i(G_1)$ sends $x \otimes 1$ to $(-1)^i x$. In particular, γ^* is onto. Therefore the above long exact sequence becomes a collection of short exact sequences

$$0 \to H^i(G) \xrightarrow{\beta^*} H^i(G_1 \sqcup \{v\}) \xrightarrow{\gamma^*} H^i(G_1) \to 0 \tag{3.1}$$

Therefore, $H^i(G) \cong \ker \gamma^*$.

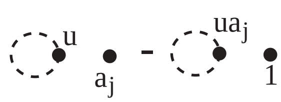
We define a homomorphism:

$$f : H^i(G_1) \otimes \mathcal{A}' \to \ker \gamma^* \text{ by } f(x \otimes a') = x \otimes a' - (-1)^i \gamma^*(x \otimes a') \otimes 1$$

One checks that f is an isomorphism of $\mathbb{Z}$-modules.

Hence $H^i(G) \cong \ker \gamma^* \cong H^i(G_1) \otimes \mathcal{A}'$. ☐

Remark 48. *The isomorphism $f : H^i(G_1) \otimes \mathcal{A}' \to H^i(G)$ defined above allows us to find generators for $H^i(G)$, namely, $e_k \otimes a_j - (-1)^i \gamma^*(e_k \otimes a_j) \otimes 1$ where e_k's form a set of generators for $H^i(G_1)$ and a_j's form a basis for $\mathcal{A}'$. This will be useful for our computations in section (3.3). The relations between the generators can also be obtained as the images of the relations under this isomorphism.*

The isomorphism can be visualized the following way. e_k is a sum of terms of the form . *This picture means that the component that includes this vertex has been assigned the value $u \in \mathcal{A}$. Such a term becomes*  *under the action of f, where the element of $\mathcal{A}$ written close to a vertex indicates the label that has been assigned to this component.*

Finally, we state a Künneth theorem type formula for our cohomology groups under disjoint union. It will be used in our computation in the next section.

Proposition 49. *For each $i \in \mathbb{N}$, we have:*

$$H^i(G_1 \sqcup G_2) \cong \left[\bigoplus_{p+q=i} H^p(G_1) \otimes H^q(G_2)\right] \oplus \left[\bigoplus_{p+q=i+1} H^p(G_1) * H^q(G_2)\right]$$

*where * denotes the torsion product of two abelian groups.*

Proof. Given our assumptions on $\mathcal{A}$, the cochain complexes for G_1 and G_2 are free. Therefore we can use the same proof as the one for Theorem (25). ☐

3.3 Some Computations

Example 50. *Let $G = N_n$ be the order n null graph. That is, the graph with n vertices and no edges. Then $C^0_{\mathcal{A}}(G) \cong \mathcal{A}^{\otimes n}$ and $C^i_{\mathcal{A}}(G) = 0$ for $i > 0$. It follows that $H^0_{\mathcal{A}}(G) \cong \mathcal{A}^{\otimes n}$ and $H^i_{\mathcal{A}}(G) = 0$ for all $i > 0$.*

Example 51. *Given any graph G, let $G \sqcup N_n$ be the graph obtained by adding n isolated vertices to G. By Proposition (49) and Example (50), $H^*(G \sqcup N_n) \cong H^*(G) \otimes \mathcal{A}^{\otimes n}$.*

Example 52. *Let $G = T_1$ be the graph with two vertices connected by an edge e. The exact sequence on (G, e) gives:*

$$0 \to H^0(G) \xrightarrow{\beta^*} H^0(G-e) \xrightarrow{\gamma^*} H^0(G/e) \xrightarrow{\alpha^*} H^1(G) \xrightarrow{\beta^*} 0$$

where $H^0(G-e) = H^0(N_2) \cong \mathcal{A} \otimes \mathcal{A}$ and $H^0(G/e) = H^0(N_1) \cong \mathcal{A}$. The map $\gamma^* : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is just the multiplication map m sending $a_1 \otimes a_2$ to $a_1 a_2$. Thus $H^0(G) \cong \ker\{m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\}$. If we assume $\mathcal{A}$ has an identity, then m is onto and therefore $H^1(G) \cong 0$. For all $i > 1$, we have $H^i(G) = 0$ by the exact sequence.

If the algebra $\mathcal{A}$ satisfies the condition in Assumptions (40), then $H^0(T_n) \cong \mathcal{A} \otimes \mathcal{A}'^{\otimes n}$.

Example 53. *In this example we compute a presentation of $H^1_{\mathcal{A}}(K_3)$ valid for any $\mathcal{A}$ as in (40) and then use this result to compute $H^1_{\mathbb{Z}[X]}(K_3)$ and $H^1_{\mathbb{Z}[X]/(X^3)}(K_3)$. We also give a presentation of $H^0_{\mathcal{A}}(K_3)$ valid for any $\mathcal{A}$.*

▲ *Let $\mathcal{A}$ be an algebra satisfying Assumptions (40). Consider the graded cochain complex for $C_{\mathcal{A}}(G)$:*

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \to 0$$

where $C^0 \cong \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, $C^1 \cong (\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{A})$, $C^2 \cong \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$, $C^3 \cong \mathcal{A}$

The cochain complex is represented in Figure (3.1).

The differentials are defined as:

$d^0(a\otimes b\otimes c) = (ab\otimes c, ac\otimes b, a\otimes bc)$

$d^1(a\otimes b, 0, 0) = (-ab, -ab, 0), d^1(0, a\otimes b, 0) = (ab, 0, -ab), d^1(0, 0, a\otimes b) = (0, ab, ab)$

$d^2(a, 0, 0) = a, d^2(0, a, 0) = -a, d^2(0, 0, a) = a.$

It follows immediately that $H^3(G) = 0$ since d^2 is onto. We also have $H^2(G) = 0$ since $\ker d^2$ is generated by $(a, a, 0)$ and $(0, a, a)$, and each of the two elements is in Imd^1. To find H^1 and H^0, we need to understand $\ker d^1$ and $Im\ d^0$.

Define $D = \{(x, x, x)|x \in \mathcal{A} \times \mathcal{A}\}, K_2 = \{(0, x, 0)|x \in \ker m\}, K_3 = \{(0, 0, x)|x \in \ker m\}$ where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map on $\mathcal{A}$. All three are subspaces of $\ker d^1$.

Claim. $\ker d^1 = D \oplus K_2 \oplus K_3$.

Proof of Claim. *Standard.*

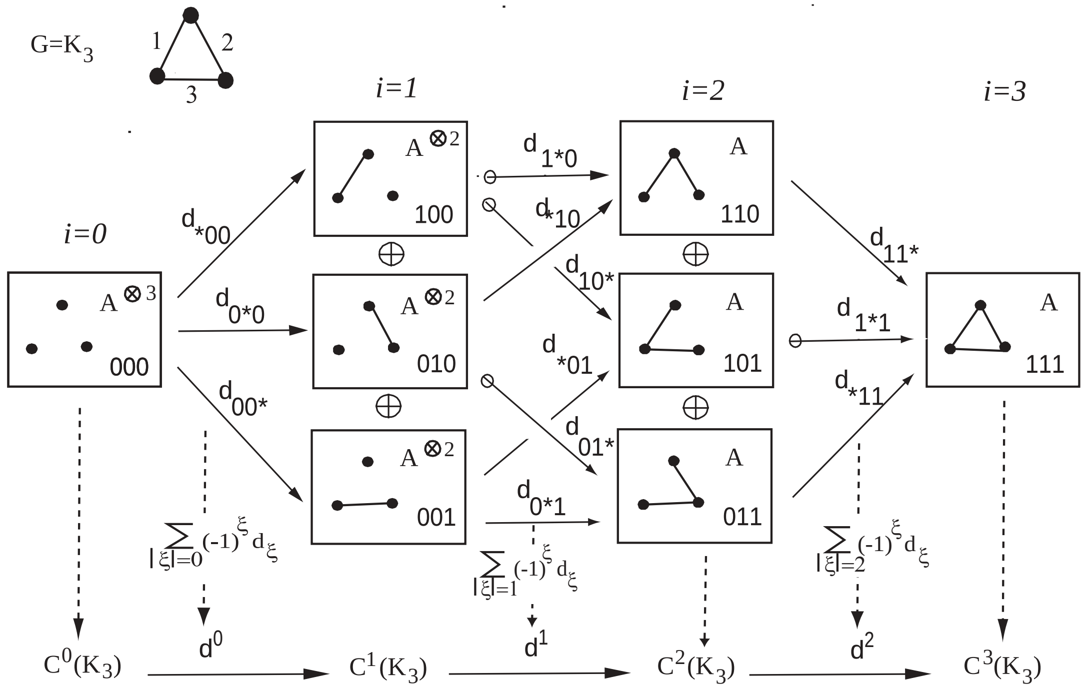


Figure 3.1: *The cochain complex $\mathcal{C}_{\mathcal{A}}(K_3)$*

Now let $\{a, b, c, \cdots, \}$ be a basis of $\mathcal{A}$. Define
$d_{a,b} = (a \otimes b, a \otimes b, a \otimes b)$,
$f_{a,b} = (0, a \otimes b - ab \otimes 1, 0)$,
$g_{a,b} = (0, 0, a \otimes b - ab \otimes 1)$. *Note that when $b = 1$, $f_{a,1} = g_{a,1} = 0$. They form a generating set of* $\ker d^1$. *We need to take* $\ker d^1$, *mod Imd^0, which is generated by all elements of the form $(ab{\otimes}c, ac{\otimes}b, a{\otimes}bc)$. A straight forward computation shows*

$$(ab{\otimes}c, ac{\otimes}b, a{\otimes}bc) = d_{ab,c} + f_{ac,b} - f_{ab,c} + g_{a,bc} - g_{ab,c}.$$

Theorem 54. *Therefore, we obtain a presentation for $H^1_{\mathcal{A}}(K_3)$ as a $\mathbb{Z}$-module:*

$$H^1_{\mathcal{A}}(K_3) \cong \{d_{a,b}, f_{a,b}, g_{a,b} | \, d_{ab,c} + f_{ac,b} - f_{ab,c} + g_{a,bc} - g_{ab,c} = 0\}$$

▲ *We now specialize to $\mathcal{A} = \mathbb{Z}[X]$. Thus $a = x^r, b = x^s, c = x^t$. For simplicity of notations, we will denote d_{x^r,x^s} by $d_{r,s}$, same for f and g. Thus $d_{r,s} = (x^r \otimes x^s, x^r \otimes x^s, x^r \otimes x^s,), f_{r,s} = (0, x^r \otimes x^s - x^{r+s} \otimes 1, 0), g_{r,s} = (0, 0, x^r \otimes x^s - x^{r+s} \otimes 1,)$. Again, if $s = 0$, we have $f_{r,0} = g_{r,0} = 0$. We show three sub-examples below, which correspond to $j = 1, 2,$ and 3 respectively.*

Example (a) $j = 1$**.** *This means the degree of each term in the generators and relations are one. For generators, we have $r + s = 1$ which implies $(r, s) = (1, 0)$ or $(0, 1)$. For* *relations, we have* $r+s+t=1$ *which implies* $(r,s,t)=(1,0,0)$, *or* $(0,1,0)$ *or* $(0,0,1)$. *This yields the following presentation of* $H^{1,1}$.

Generators: $d_{1,0}, d_{0,1}, f_{1,0}, f_{0,1}, g_{1,0}, g_{0,1}$, *where* $f_{1,0}=g_{1,0}=0$

Relations: $d_{1,0}+f_{1,0}-f_{1,0}+g_{1,0}-g_{1,0}=0$,

$d_{1,0}+f_{0,1}-f_{1,0}+g_{0,1}-g_{1,0}=0$,

$d_{0,1}+f_{1,0}-f_{0,1}+g_{0,1}-g_{0,1}=0$.

This implies that $H^{1,1}\cong\mathbb{Z}$ *with generator being* $f_{0,1}=(0,1\otimes x-x\otimes 1,0)$.

Example (b) $j=2$. *The solutions for* $r+s=2$ *are* $(2,0),(1,1),(0,2)$. *The solutions for* $r+s+t=2$ *are* $(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)$. *We have the following presentation for* $H^{1,2}$,

Generators: $d_{2,0}, d_{1,1}, d_{0,2}, f_{1,1}, f_{0,2}, g_{1,1}, g_{0,2}$. *(this time we dropped* $f_{1,0}$ *and* $g_{1,0}$ *which are 0.*

Relations: $d_{2,0}=0$

$d_{2,0}+f_{0,2}-0+g_{0,2}-0=0$

$d_{0,2}+0-f_{0,2}+g_{0,2}-g_{0,2}=0$

$d_{2,0}+f_{1,1}-0+g_{1,1}-0=0$

$d_{1,1}+0-f_{1,1}+g_{1,1}-g_{1,1}=0$

$d_{1,1}+f_{1,1}-f_{1,1}+g_{0,2}-g_{1,1}=0$.

This implies $H^{1,2}\cong\mathbb{Z}$, *generated by* $f_{1,1}$. *Other generators can be expressed in terms of* $f_{1,1}$ *as follows:* $d_{2,0}=0, d_{1,1}=f_{1,1}, d_{0,2}=f_{0,2}=-g_{0,2}=2f_{1,1}, g_{1,1}=-f_{1,1}$.

Example (c)$j=3$. *The solutions for* $r+s=3$ *are* $(3,0),(2,1),(1,2),(0,3)$. *The solutions for* $r+s+t=3$ *are:*

$(3,0,0),(0,3,0),(0,0,3),(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2),(1,1,1)$. *Thus a presentation for* $H^{0,3}$ *is:*

Generators: $d_{3,0}, d_{2,1}, d_{1,2}, d_{0,3}, f_{2,1}, f_{1,2}, f_{0,3}, g_{2,1}, g_{1,2}, g_{0,3}$.

Relations: $d_{3,0}+0-0+0-0=0$

$d_{3,0}+f_{0,3}-0+g_{0,3}-g_{3,0}=0$

$d_{0,3}+0-f_{0,3}+g_{0,3}-g_{0,3}=0$

$d_{3,0}+f_{2,1}-0+g_{2,1}-0=0$

$d_{3,0}+f_{1,2}-0+g_{1,2}-0=0$

$d_{2,1}+0-f_{2,1}+g_{2,1}-g_{2,1}=0$

$d_{1,2}+0-f_{1,2}+g_{1,2}-g_{1,2}=0$

$d_{2,1}+f_{1,2}-f_{2,1}+g_{0,3}-g_{2,1}=0$

$d_{1,2}+f_{2,1}-f_{1,2}+g_{0,3}-g_{1,2}=0$

$d_{2,1}+f_{2,1}-f_{2,1}+g_{1,2}-g_{2,1}=0$

This implies that $H^{1,3}\cong\mathbb{Z}$ *with* $f_{2,1}$ *being a generator. Other generators can be expressed in terms of* $f_{2,1}$ *as follows:* $d_{3,0}=0, d_{2,1}=f_{2,1}, d_{1,2}=2f_{2,1}, d_{0,3}=3f_{2,1}, f_{1,2}=2f_{2,1}, f_{0,3}=$

$3f_{2,1}, g_{2,1} = -f_{2,1}, g_{1,2} = -2f_{2,1}, g_{0,3} = -3f_{2,1}$.

▲ *This allows us to compute cohomology groups of K_3 when $\mathcal{A} = \mathbb{Z}[X]/(X^k)$. For example, let $k = 3$, we have $X^3 = 0$ in $\mathcal{A}$. Thus $d_{0,3} = f_{0,3} = g_{0,3} = 0$ (we already knew that $d_{3,0} = f_{3,0} = g_{3,0} = 0$. Thus the above presentation implies the following presentation of $H^{1,3}(C_3)$ when $\mathcal{A} = \mathbb{Z}[X]/(X^3)$:*
Generator: $f_{2,1}$, relations: $3f_{2,1} = 0$ this yields $H^{1,3}(K^3) \cong \mathbb{Z}_3$.
Recall that when $\mathbb{Z}[X]/(X^2)$, $H^{1,2}(K_3) \cong Z_2$.

▲ *These computations also show the induced homomorphism $G(f) : H_{\mathcal{A}}(K_3) \to H_{\mathcal{B}}(K_3)$ when $f : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism. For example, if $f : \mathbb{Z}[X] \to \mathbb{Z}[X]/(X^3)$ is the obvious onto homomorphism, then $G(f) : H^{1,3}_{\mathcal{A}}(K_3) \to H^{1,3}_{\mathcal{B}}(K_3)$ is the homomorphism sending $f_{2,1}$ in the domain to $f_{2,1}$ in the target. In other words, it is the homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_3$ sending 1 to 1.*

▲ *Finally, for $H^0(G) = \ker d^0$ which is the set of all elements of the form*
$\sum_{a,b,c} \lambda_{a,b,c} a \otimes b \otimes c$ that satisfy the system of equations:
$\sum_{a,b,c} \lambda_{a,b,c} ab \otimes c = 0$
$\sum_{a,b,c} \lambda_{a,b,c} ac \otimes b = 0$
$\sum_{a,b,c} \lambda_{a,b,c} a \otimes bc = 0$
Specific computations involve some standard linear algebra.

Example 55. *Let $G = K_3$, the circuit graph with 3 vertices and 3 edges. Let $\mathbb{Z}[X]/(X^3)$, we have*

$$H^0(C_3) \cong \mathbb{Z}\{3\} \oplus \mathbb{Z}^3\{4\} \oplus \mathbb{Z}^3\{5\} \oplus \mathbb{Z}\{6\}$$

$$H^1(G) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}_3\{3\}$$

and $H^i(G) = 0$ for $i > 1$.

We are going to show the computation using the exact sequence.

◇ If $G = N_1$, the graph with one vertex and no edge, then $H^0(G) \cong \mathcal{A}$ with generators being $\{x^r | r = 0, 1, 2\}$.

◇ If $G = T_1$, the tree with one edge, Corollary (47) and Remark (48) implies that $H^0(G) \cong \mathcal{A} \otimes \mathcal{A}'$ with generators $\{e_{rs} | r = 0, 1, 2, s = 1, 2\}$, where $e_{rs} = x^r \otimes x^s - x^{r+s} \otimes 1$ in $H^0(G)$. e_{rs} can be seen as $\overset{x^r}{\bullet}\ \overset{x^s}{\bullet} - \overset{x^{r+s}}{\bullet}\ \overset{1}{\bullet}$. For simplicity, we denote e_{rs} by $e_{rs} = x^r x^s - x^{r+s} 1$ (without the tensor product symbols).

◇ If $G = T_2$, the tree with two edges, the same argument shows that a set of generators of $H^0(G) \cong H^0(T_1) \otimes \mathcal{A}'$ is $\{e_{rst} | r = 0, 1, 2, s = 1, 2$ and $t = 1, 2\}$ where

$e_{rst} = (x^r x^s x^t - x^r x^{s+t} 1) - (x^{r+s} 1 x^t - x^{r+s} x^t 1)$ and can be represented by

⋄ Now, let $G = K_3$ be the circuit graph with three vertices and three edges. Let e be an edge of K_3. The exact sequence on (K_3, e) gives

$$0 \to H^0(K_3) \xrightarrow{\beta^*} H^0(K_3 - e) \xrightarrow{\gamma^*} H^0(K_3/e) \xrightarrow{\alpha^*} H^1(K_3) \to 0$$

where $H^0(K_3 - e) = H^0(T_2)$ is freely generated by
$\{e_{011}, e_{012}, e_{021}, e_{111}, e_{022}, e_{112}, e_{121}, e_{211}, e_{122}, e_{212}, e_{221}, e_{222}\}$,
and $H^0(K_3/e) \cong H^0(T_1)$ is freely generated by $\{e_{01}, e_{02}, e_{11}, e_{12}, e_{21}, e_{22}\}$. The map γ^* sends $a \otimes b \otimes c$ to $ac \otimes b$. Thus $\gamma^*(e_{rst}) = x^{r+t}x^s - x^r x^{s+t} - x^{r+s+t}1 + x^{r+s}x^t = (x^{r+t}x^s - x^{r+s+t}1) - (x^r x^{s+t} - x^{r+s+t}1) + (x^{r+s}x^t - x^{r+s+t}1) = e_{r+t,s} - e_{r,s+t} + e_{r+s,t}$. This gives

$\gamma^*(e_{011}) = e_{11} - e_{02} + e_{11} = 2e_{11} - e_{02}$

$\gamma^*(e_{012}) = e_{21} - e_{03} + e_{12} = e_{12} + e_{21}$ (here $e_{03} = 0$ since $x^3 = 0$)

$\gamma^*(e_{021}) = e_{12} - e_{03} + e_{21} = e_{12} + e_{21}$

$\gamma^*(e_{111}) = e_{21} - e_{12} + e_{21} = -e_{12} + 2e_{21}$

$\gamma^*(e_{022}) = e_{22} - e_{04} + e_{22} = 2e_{22}$

$\gamma^*(e_{112}) = e_{31} - e_{13} + e_{22} = e_{22}$

$\gamma^*(e_{121}) = e_{22} - e_{13} + e_{31} = e_{22}$

$\gamma^*(e_{211}) = e_{31} - e_{22} + e_{31} = -e_{22}$

$\gamma^*(e_{122}) = e_{32} - e_{14} + e_{32} = 0$

$\gamma^*(e_{212}) = e_{41} - e_{23} + e_{32} = 0$

$\gamma^*(e_{221}) = e_{32} - e_{23} + e_{41} = 0$

$\gamma^*(e_{222}) = e_{42} - e_{24} + e_{42} = 0$

This allows us to compute the cohomology groups of K_3, since $H^0(K_3) \cong \ker \gamma^*$, $H^1(K_3) \cong H^0(K_3/e)/\mathrm{Im}\gamma^*$. Although the computation involves a 12×6 matrix, the actual computation is quite simple, since the map γ^* breaks into several maps according the degree (i.e. $r + s + t$ in the domain).

The same idea can be applied to the ring $\mathcal{A} = \mathbb{Z}[X]$. However, in this case, the "thickness" of the cohomology groups is infinite. In other words, there are infinitely many j such that $H^{0,j}(K_3)$ is nonzero. We have the following partial result:
$H^{0,0}(K_3) = H^{0,1}(K_3) = H^{0,2}(K_3) = H^{0,3}(K_3) = 0$
For $j \geqslant 4$, $H^{0,j}(K_3)$ is nonzero.

A pattern seems to appear: We define the $\mathbb{Z}$-algebra $\mathcal{A}_m$ by $\mathcal{A}_m := \mathbb{Z}[X]/X^m$. Our examples show that $H^{1,2}_{\mathcal{A}_2}(P_3)) = \mathbb{Z}_2$ and $H^{1,3}_{\mathcal{A}_3}(P_3)) = \mathbb{Z}_3$. We make the following conjecture:

Conjecture 56. *Let K_3 be the polygon graph on 3 vertices,*

$$H^{1,m}_{\mathcal{A}_m}(P_3) = \mathbb{Z}_m$$

This dissertation was defended in May 2005 and the above result has since then been proved in [HPR05] in July 2005.

Ring with no grading

Next, let us consider the special case when our ring $\mathcal{A}$ has no grading, i.e. every element of $\mathcal{A}$ has degree 0. The graded dimension of $\mathcal{A}$ is then an integer, namely $\dim \mathcal{A}$. Thus the Euler characteristic of our cohomology groups is the integer $P_G(\lambda)$ where $\lambda = \dim \mathcal{A}$.

We would like to mention the work of Eastwood and Huggett in [EH00]. For each positive integer λ and each graph G, they construct a topology space M whose Euler characteristic is the integer $P_G(\lambda)$. We don't know if there is any connection between the cohomology groups of M and the cohomology groups here.

▲ *Ring with no grading, rank($\mathcal{A}$)=1*

Example 57. *Let $\mathcal{A} = \mathbb{Z}$ with the usual ring structure. We have $q\dim \mathcal{A} = 1$, $P(G, 1) = 1$ if G has no edge and 0 otherwise. For the cohomology groups, we have*

$$H^i(G) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ and } G \text{ has no edge,} \\ 0 & \text{otherwise.} \end{cases}$$

This can be easily proved by inducting on the number of edges and using the long exact sequence.

▲ *Ring with no grading, rank($\mathcal{A}$)=2*

Next, we will compute a few examples when $\dim \mathcal{A} = 2$. Thus $(\mathcal{A}, +)$ is the free abelian group of rank two, with generators 1 and x. Let us consider various ring structures on $\mathcal{A}$ so that 1 is the identity. Therefore the product $*$ satisfies $1 * 1 = 1, 1 * x = x * 1 = x$, and $x * x = a1 + bx$ where a, b are two fixed integers. In the Annexes, in section (6.2), we show that the isomorphism type of such a ring depends on $a^2 + 4b$.

Example 58. *Let $\mathcal{A}$ be the ring above. Then*

$$H^0_{\mathcal{A}}(K_3) \cong \begin{cases} \mathbb{Z} & \text{if } b^2 + 4a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H^1_{\mathcal{A}}(K_3) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z} & \text{if } b^2 + 4a = 0, \\ \mathbb{Z}_{|b^2+4a|} & \text{if } b^2 + 4a \neq 0 \text{ and } b \text{ is odd} \\ \mathbb{Z}_2 \oplus Z_{\frac{|b^2+4a|}{2}} & \text{if } b^2 + 4a \neq 0 \text{ and } b \text{ is even} \end{cases}$$

$H^i_{\mathcal{A}}(G) = 0$ *for all* $i > 1$.

The computation is based on the exact sequence similar to last example.

⋄ If $G = N_1$, the graph with one vertex and no edge, then $H^0(G) \cong \mathcal{A}$ with generators being 1 and x.

$\diamond$ If $G = T_1$, the tree with one edge, Corollary (47) implies that $H^0(G) \cong \mathcal{A} \otimes \mathbb{Z}x \cong \mathcal{A}$. By Remark (48), we have the following basis: $e_1 = 1 \otimes x - x \otimes 1 = 1x - x1, e_2 = x \otimes x - x * x \otimes 1 = x \otimes x - (a1 + bx) \otimes 1 = x \otimes x - a1 \otimes 1 - bx \otimes 1 = xx - a11 - bx1$. For simplicity of notation, we suppressed the tensor product symbol $\otimes$. Thus $1x$ denotes $1 \otimes x$.

$\diamond$ If $G = T_2$, the tree with two edges, the same argument shows that $H^0(T_2) \cong \mathcal{A}$. To describe the basis, we denote the three vertices of T_2 by 1, 2, and 3 with 2 being the vertex with degree two. A basis is:

$f_1 = (1 \otimes x \otimes x - 1 \otimes x * x \otimes 1) - (x \otimes 1 \otimes x - x \otimes 1 * x \otimes 1)$
$= (1xx) - 1(a1 + bx)1 - (x1x) + (xx1) = (1xx) - a(111) - b(1x1) - (x1x) + xx1),$
$f_2 = (xxx) - [x(x * x)1] - a(11x - 1x1) - b(x1x - xx1)$
$= (xxx) - x(a1 + bx)1 - a(11x) + a(1x1) - b(x1x) + b(xx1)$
$= (xxx) - a(x11) - b(xx1) - a(11x) + a(1x1) - b(x1x) + b(xx1).$

$\diamond$ Now, let $G = K_3$ be the circuit graph with three vertices and three edges. Let e be an edge of K_3. The exact sequence on (K_3, e) gives

$$0 \to H^0(K_3) \xrightarrow{\beta^*} H^0(K_3 - e) \xrightarrow{\gamma^*} H^0(K_3/e) \xrightarrow{\alpha^*} H^1(K_3) \to 0$$

We have $H^0(K_3 - e) = H^0(T_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $\{f_1, f_2\}$ being a basis, and $H^0(K_3/e) \cong H^0(T_1) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $\{e_1, e_2\}$ being a basis. The map γ^* adds the edge e to T_2, adjust the color using the multiplication on $\mathcal{A}$, and then shrink e to a point. In other words, it sends $a \otimes b \otimes c$ to $ac \otimes b$. Thus $\gamma^*(f_1) = \gamma^*((1xx) - a(111) - b(1x1) - (x1x) + xx1) = xx - a11 - b1x - (x^2)1 + xx = 2(xx) - a(11) - b(1x) - (a1 + bx)1 = 2[(xx) - a(11) - b(x1)] - [b(1x) - b(x1)] = -be_1 + 2e_2$. $\gamma^*(f_2) = \gamma^*((xxx) - a(x11) - b(xx1) - a(11x) + a(1x1) - b(x1x) + b(xx1))$
$= (x * x)x - a(x1) - b(xx) - a(x1) + a(1x) - b((x * x)1) + b(xx)$
$= (a1 + bx)x - 2a(x1) - b(xx) + a(1x) - b((a1 + bx)1) + b(xx)$
$= a(1x) + b(xx) - 2a(x1) - b(xx) + a(1x) - ab(11) - b^2(x1) + b(xx)$
$= b(xx) - 2a(x1) - b^2(x1) + 2a(1x) - ab(11)$
$= 2a(1x - x1) + b(xx) - ab(11) - b^2(x1)$
$= 2ae_1 + be_2$

If $b^2 + 4a = 0$, $H^0(K_3) \cong \mathbb{Z}$. Otherwise, $H^0(K_3) = 0$.

We now determine H^1. $H^1(K_3)$ has a presentation with generators e_1, e_2 and relations $-be_1 + 2e_2 = 0$, and $2ae_1 + be_2 = 0$.

If b is odd, then $b^2 + 4a \neq 0$, and we have $H^1(K_3) \cong \mathbb{Z}_{|b^2+4a|}$.

If b is even, then $H^1(K_3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ if $b^2 + 4a = 0$, and $H^1(K_3) \cong \mathbb{Z}_{\frac{|b^2+4a|}{2}} \oplus \mathbb{Z}_2$ if $b^2 + 4a \neq 0$.

This implies the announced results for the cohomology groups.

Example 59. ***A sequence of graphs with the same chromatic polynomial that can be distinguished by their cohomology groups, provided we allow a larger class of differential, as explained in Remark (43).***

Let L_n be the graph with one vertex and n edges. For $n \geqslant 1$, all these graph have at least one loop so their chromatic polynomial is zero.

Let $\mathcal{A} = \mathbb{Z}X$ where X has degree 1 and satisfies $X^2 = 0$. We set the per edge map corresponding to adding the edge e to be the multiplication in $\mathcal{A}$ if adding e decreases the number of components and to be 0 otherwise, as allowed by Remark (43). This produces a differential that is always equal to zero, hence the cohomology groups are the cochain groups.

It suffices to show that the cochain groups distinguish these graphs. But this is easy to see since,

$$\text{For } n \geqslant 1, C^i(L_n) = \begin{cases} \mathcal{A} & \text{if } 0 \leqslant i \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Tutte polynomial distinguishes these graphs since $t(L_n) = y^n$ for $n \geq 1$.

Chapter 4

Determine which graphs have torsion in at least one cohomology group

The results of this section are covered in [HPR05].

A natural question is to determine under which circumstances at least one of the cohomology group has a torsion part. The main result is that, when the algebra on which the construction is based is $\mathbb{Z}[X]/(X^2)$, a loopless graph will always have a $\mathbb{Z}_2$ torsion provided it contains a cycle of length at least 3.

In order to alleviate the writing, we might simply say $H^*(G)$ contains a torsion to mean that at least one of the $H^*(G)$ contains a torsion.

4.1 When the algebra is $\mathbb{Z}[X]/(X^2)$

4.1.1 Facts and observations

▲ We start by reminding some facts that were proved earlier and that are going to be useful here.

Fact 60. *The cohomology groups of a forest, so in particular the ones of a tree, don't have torsion.*

Fact 61. *If the graph has a loop then all the cohomology groups are trivial.*

Fact 62. *The cohomology groups are unchanged if the multiple edges of a graph are replaced by single edges.*

▲ This table has been copied from computational results for circuit graphs (35). It illustrates our computational results (up to $n = 6$ and $i = 4$) for polygonal graphs (= circuit graphs), where n is the length of the cycle and i is the height of the cohomology group.

$n\backslash i$	H^0	H^1	H^2	H^3	H^4
P_1	0	0	0	0	0
P_2	$\mathbb{Z}\{2\} \oplus \mathbb{Z}\{1\}$	0	0	0	0
P_3	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	0	0	0
P_4	$\mathbb{Z}\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	0	0
P_5	$\mathbb{Z}\{5\}$	$\mathbb{Z}_2\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$	0
P_6	$\mathbb{Z}\{6\} \oplus \mathbb{Z}\{5\}$	$\mathbb{Z}\{5\}$	$\mathbb{Z}_2\{4\} \oplus \mathbb{Z}\{3\}$	$\mathbb{Z}\{3\}$	$\mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}$

We first note that, for all $n \geq 3$, $H^*(P_n)$ contains torsion.

A closer look at these examples reveals that, in the case of an odd cycle, there seems to always be a torsion in $H^1(G)$ in degree $p-1$, and in the case of an even cycle, there seems to always be a torsion in $H^2(G)$ in degree $p-2$ where p is the number of vertices of the graph. These remarks will guide us for the formulation of the lemmas.

4.1.2 The result

Theorem 63. *Let G be a graph. We denote by G' the graph obtained from G by replacing multiple edges by single edges. The following are equivalent.*

1. *$H^*(G)$ contains a torsion part,*
2. *$H^*(G)$ contains a $\mathbb{Z}_2$-torsion,*
3. *G has no loops and G' is* NOT *a forest, or, equivalently, G has no loops and contains a cycle of order ≥ 3.*

Note that this doesn't mean that $\mathbb{Z}_2$ is the only possible torsion. It only implies that if there is a torsion different from $\mathbb{Z}_2$ then there will also be $\mathbb{Z}_2$-torsion.

Proof. (2) $\Rightarrow$ (1) is obvious.
(1) $\Rightarrow$ (3) Let ℓ be the maximum length of a cycle in G.
ℓ can't be 0 because that would mean that G is a forest. This would contradict Fact (60).
ℓ can't be 1 because that would mean that G has a loop. This would contradict Fact (61).
Assume $\ell = 2$. That means that G has a double edge. Hence, the maximum length of a cycle in G' would be 0 or 1. As we just saw, this means that the cohomology groups of G' have no torsion. However, G' and G have the same cohomology groups by Fact (62). This contradicts the assumption (1).
(3) $\Rightarrow$ (2) is what we are going to prove below in Lemmas (64) and (65). □

4.1.3 The odd cycle case

Lemma 64. *If a loopless graph G with p vertices contains an odd cycle of length ≥ 3, then $H^{1,p-1}(G)$ contains a $\mathbb{Z}_2$-torsion.*

Proof. Let G be a loopless graph with p vertices containing a cycle of length $2s+1$ with $s \geq 1$.

It suffices to find an element z in $\ker d^1$ of degree $p-1$ such that $2z = 0$ in $H^1(G)$ and $z \neq 0$ in $H^1(G)$. The condition $2z = 0$ in $H^1(G)$ is the same as $2z \in \mathrm{Im} d^{0,p-1}$. Therefore, our first step will be to determine $\mathrm{Im} d^{0,p-1}$.

▲ Matrix representation of $d^{0,p-1}$:

For convenience, we label the vertices of the graph starting with the ones in the cycle. The vertices in the cycle are labeled monotonically v_1 to v_{2s+1}, with the requirement that each v_i is adjacent to v_{i+1} and v_{2s+1} is adjacent to v_1. The vertices that are not in the cycle are labeled v_{2s+2} to v_p. Examples are given in Figure (4.1).

We label the edges in the cycle so that e_i is the edge $v_i v_{i+1}$ for $1 \leq i \leq 2s$ and e_{2s+1} is the edge $v_{2s+1}v_1$. The edges that are not in the cycle are labeled e_{2s+2} to e_n.

The basis elements of $C^{0,p-1}(G)$ have p components and degree $p-1$ which means that all vertices are assigned the value X except one which is assigned the value 1. The basis element for which 1 is assigned to the vertex v_i and with X assigned to all the other vertices is denoted by b_i. The basis elements for which the 1 is assigned to a vertex in the cycle are b_1 to b_{2s+1}.

In order to write a matrix for $d^{0,p-1}$, we also need to describe the basis elements of the target space $C^{1,p-1}(G)$. First, note that each of these basis elements contains one edge, which, by assumption, is not a loop. Thus each of these basis elements has $p-1$ components. Therefore for degree reasons, all the components are assigned the value X. The basis element for which the present edge is e_i and with X assigned to all the components is denoted by a_i. The basis elements for which the present edge is in the cycle are a_1 to a_{2s+1}.

With these notations, we get a matrix representing $d^{0,p-1}$, which is shown in (4.1) below

.

▲ Let $z = \sum_{i=1}^{n} a_i$

△ We first prove that $2z \in \mathrm{Im} d^{0,p-1}$:

In the matrix representation of $d^{0,p-1}$, the coordinates of the image of b_i under the differential, in the basis $(a_i)_i$ are listed in the i^{th} row.

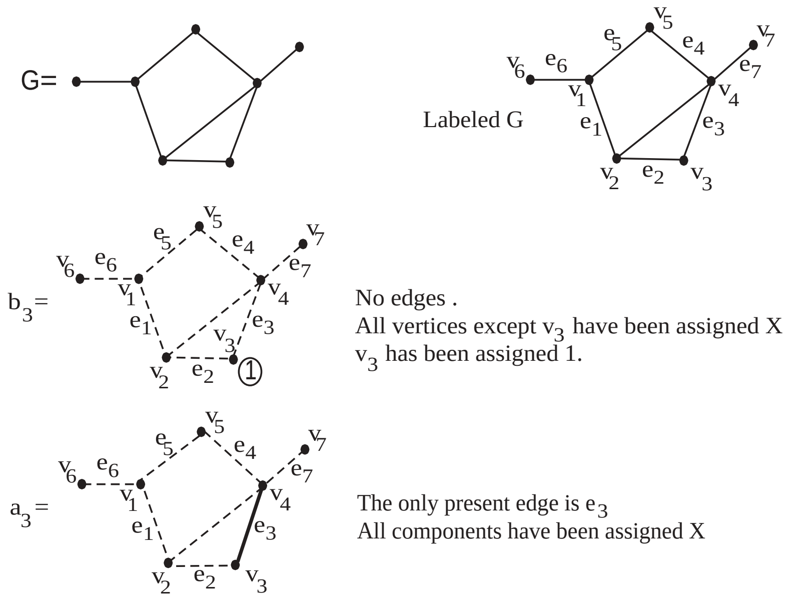


Figure 4.1: *Notation for basis elements basis element in* $C^{0,p-1}(G)$ *and* $C^{1,p-1}(G)$

$$
\begin{array}{rcc}
\textit{Matrix of } d^{0,p-1} & \begin{array}{ccccc|ccc} a_1 & a_2 & \cdots & & a_{2s+1} & a_{2s+2} & \cdots & a_n \end{array} & \text{Coeff} \\
\begin{array}{r} d(b_1)\to \\ d(b_2)\to \\ \vdots \\ \vdots \\ d(b_{2s+1})\to \\ ---- \\ d(b_{2s+2})\to \\ \vdots \\ d(b_p)\to \end{array} &
\left[\begin{array}{ccccc|ccc}
1 & & & & 1 & & & \\
1 & 1 & & & & & & \\
 & \ddots & \ddots & & & & M_1 & \\
 & & 1 & 1 & & & & \\
 & & & 1 & 1 & & & \\
- & - & - & - & - & - & - & - \\
 & & & & & & & \\
 & & 0 & & & & M_2 & \\
 & & & & & & &
\end{array}\right] &
\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ -- \\ 1 \\ \vdots \\ 1 \end{array} \\
\hline
d\left(\sum b_i\right) & \underbrace{\begin{array}{ccccc|ccc} 2, & 2, & \cdots & \cdots & 2 & 2, & \cdots\ \cdots & ,2 \end{array}}_{=2z} &
\end{array}
\tag{4.1}
$$

The coefficients (all equal to 1 in (4.1)) in the column at the right indicate the coefficient by which each line is multiplied before addition.

By adding all the rows of this matrix, we see $d\left(\sum_{i=1}^{p} b_i\right) = 2z$ so $2z \in \mathrm{Im} d^{0,p-1}$.

Note that since $C^2(G)$ doesn't have torsion, this implies that $z \in \ker d^{1,p-1}$. Indeed, $d^1(2z) = 0$ so $2d^1(z) = 0$ in $C^2(G)$.

The reason why adding the rows of M_1 and M_2 always yields a coordinate equal to 2 on each $\{a_j\}_{j\geq 2s+2}$ is that each edge has two ends (remember that there are no loops) so each of these $\{a_j\}_{j\geq 2s+2}$ is in the image of exactly two b_i's, the ones corresponding to a 1 placed at each endpoint of the edge e_j.

$\triangle$ It remains to show that $z \notin \mathrm{Im} d^{0,p-1}$. Assume $z = d(x)$ for some x in $C^{0,p-1}$. This x can be written $x = \sum_{i=1}^{p} \alpha_i b_i$ for some α_i.

$$
\begin{array}{c|c|c}
\textit{Matrix of } d^{0,p-1} & \begin{array}{ccccc|ccc} a_1 & a_2 & \cdots & & a_{2s+1} & a_{2s+2} & \cdots & a_n \end{array} & \text{Coeff} \\
\begin{array}{c} d(b_1) \to \\ d(b_2) \to \\ \vdots \\ \vdots \\ d(b_{2s+1}) \to \\ ---- \\ d(b_{2s+2}) \to \\ \vdots \\ d(b_p) \to \end{array} &
\left[\begin{array}{ccccc|c}
1 & & & & 1 & \\
1 & 1 & & & & \\
 & \ddots & \ddots & & & M_1 \\
 & & 1 & 1 & & \\
 & & & 1 & 1 & \\
\hline
 & & & & & \\
 & & 0 & & & M_2 \\
 & & & & &
\end{array}\right] &
\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{2s} \\ \alpha_{2s+1} \\ -- \\ \alpha_{2s+2} \\ \vdots \\ \alpha_p \end{array} \\
\hline
z = d\left(\sum_{i=1}^{p} \alpha_i b_i\right) & \underbrace{1, \ 1, \ \cdots \ \cdots \ 1 \mid 1, \ \cdots \ \cdots \ 1}_{=z} &
\end{array}
$$

This means that the result of multiplying the first line by α_1, the second line by α_2, etc, and adding all the lines yields $(1, 1, \cdots, 1)$, the coordinates of z on the a_i's.

If we now read this by columns, we get a contradiction:

$$
\begin{array}{rccccccccl}
\text{Column 1:} & \alpha_1 & + & \alpha_2 & & & & & & = 1 \\
+\text{Column 2:} & & & \alpha_2 & + & \alpha_3 & & & & = 1 \\
\vdots & & & & & \ddots & \ddots & & & \vdots \\
 & & & & & & \alpha_{2s} & + & \alpha_{2s+1} & \vdots \\
+\text{Column 2s+1:} & \alpha_1 & + & & & & & + & \alpha_{2s+1} & = 1 \\
\hline
 & \multicolumn{8}{c}{\underbrace{2(\alpha_1 \quad + \quad \alpha_2 \qquad \alpha_{2s} \quad + \quad \alpha_{2s+1})}_{\text{even}}} & = \underbrace{2s+1}_{\text{odd}}
\end{array}
$$

This shows that $z \notin \mathrm{Im} d^{0,p-1}$.

$\square$

4.1.4 The even cycle case

The simplest simple graph without an odd cycle is the "square" C_4. As mentioned earlier, in this case the torsion appears only for $H^2(G)$, which indicates that we have to look "deeper" into the cohomology to find torsion than in the odd cycle case.

Lemma 65. *Let G be a simple graph, i.e. a graph with no loops and no multiple edges. If G contains an even cycle of length ≥ 4, then $H^{2,p-2}(G)$ contains a $\mathbb{Z}_2$-torsion.*

Proof. It suffices to find an element z in $\ker d^{2,p-2}$ such that $2z = 0$ in $H^2(G)$ and $z \neq 0$ in $H^2(G)$.

$2z = 0$ in $H^2(G)$ means that $2z \in \mathrm{Im} d^{1,p-2}$ so our first step will be to determine $\mathrm{Im} d^{1,p-2}$.

▲ *Matrix representation of $d^{1,p-2}$:*

The labelling of the vertices and the edges of the graph is the same as described in the odd cycle case, as illustrated in Figure (4.1). The basis elements of $C^{1,p-2}(G)$ have $p-1$ components since one edge is present and there are no loops. They have degree $p-2$ which means that all components are assigned the value X except one that is assigned the value 1. The basis element for which the present edge is e_i and the vertex that is assigned 1 is v_j is denoted by b_i^j. An example is given in Figure (4.2).

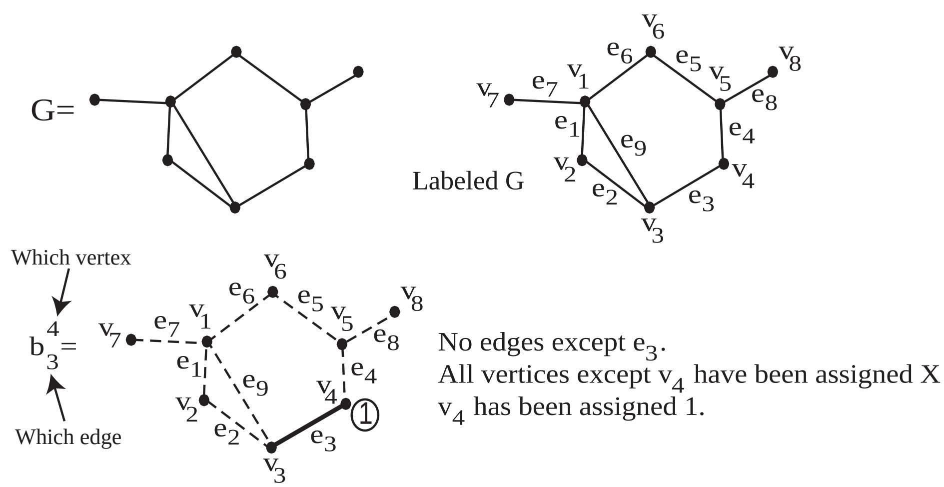


Figure 4.2: *Notation for basis elements basis element in $C^{1,p-2}(G)$*

We also need to describe the basis elements of the target space $C^{2,p-2}(G)$. First, note that each has $p-2$ components because, since there are no loops and no multiple edges, adding two edges automatically decreases the number of components by two. Therefore for degree reasons, all the components are assigned the color X. The basis element for which the present edges are e_i and e_j, with X assigned to all the components is denoted by a_{ij} with $i < j$.

With these notations, we get a matrix representing $d^{1,p-2}$, which is shown in the next paragraph.

▲ We are now ready to exhibit an element z in $\ker d^2$ of degree $p-2$ such that $2z = 0$ in $H^2(G)$ and $z \neq 0$ in $H^2(G)$.

△ We first prove that there exists an element in $\mathrm{Im} d^{1,p-2}$ with all coordinates even. This is the $2z$ we were looking for.

In the matrix representation of $d^{1,p-2}$, the coordinates of the image of b_i^j under the differential, in the basis $(a_{ij})_{i,j}$ are listed in the i^{th} row.

$$
\begin{array}{rcc}
\textit{Matrix of } d^{1,p-2} & \overbrace{a_{12}\quad a_{13}\quad \cdots \quad a_{1,2s}}^{e_1 \text{ and another edge in cycle}} \;\Big|\; \overbrace{a_{23}\quad \cdots \quad a_{2s-1,2s}}^{e_i,\, e_j \text{ in cycle},\ i,j\neq 1} \;\Big|\; \overset{\text{At least one edge not in cycle}}{\text{all other } a_{ij}} & \text{Coeff} \\
\begin{array}{r}
d(b_1^1) \\ d(b_1^3) \\ \\ \\ d(b_1^{2s}) \\ \hline
d(b_1^{2s+1}) \\ \vdots \\ d(b_1^p) \\ \hline
d(b_i^j) \\ e_i,\, v_j \text{ in cycle} \\ i\neq 1 \\ \hline
\\ \text{other } b_i^j \\ \\
\end{array}
&
\left[\begin{array}{ccccc|c|c}
-1 & & & & -1 & & \\
-1 & -1 & & & & & \\
 & -1 & \ddots & & & 0 & M_1 \\
 & & \ddots & -1 & & & \\
 & & & -1 & -1 & & \\ \hline
 & & & & & & \\
 & & 0 & & & 0 & M_2 \\
 & & & & & & \\ \hline
 & & & & & & \\
 & & M_3 & & & M_4 & M_5 \\
 & & & & & & \\ \hline
 & & & & & & \\
 & & 0 & & & 0 & M_6 \\
 & & & & & & \\
\end{array}\right]
&
\begin{array}{c}
1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \hline
1 \\ \vdots \\ 1 \\ \hline
0 \\ \vdots \\ 0 \\ \hline
0 \\ \vdots \\ 0 \\
\end{array}
\\
d\left(\sum_{1\leq j\leq p} b_1^j\right) = & -2 \quad -2 \quad \cdots \quad \cdots \quad -2 \;\Big|\; 0 \quad \cdots \quad 0 \;\Big|\; -2\varepsilon_{ij} & = 2z
\end{array}
$$

where $\varepsilon_{ij} \in \{0,1\}$ for all $(i,j) \in J$ where J is the set of all (i,j) such that at least one of the edges e_i, e_j is not in the cycle and $i < j$.

"Other b_{ij}" means either e_i not in the cycle or e_i is in the cycle but v_j isn't.

We need to explain why adding the rows of M_1 and M_2 always yields a coordinate equal to -2 or 0 on each a_{ij}, $(i,j) \in J$. If $i = 1$, each of these a_{1j} is in the image of exactly two b_1^j's, the ones corresponding to a 1 placed at each endpoint of the edge e_j (since each edge has two ends under the assumption that there are no loops). If $i \neq 1$, a_{ij} is in the image of none of the b_1^j's hence the coordinate on the a_{ij} with $i \neq 1$ are 0.

It remains to explain the negative signs. Each b_1^j is a basis element coming from the state labeled 10....0 (the present edge is the first one in the ordering) so the label of the per-edge map that adds the edge i with $i \geq 2$ is $10..0*0..0$ with the star in the i^{th} position. The definition of the differential says that when there is an odd number of 1's before the star, the map is assigned a negative sign.

By adding the p first rows of this matrix, we see that all the coordinates of $d\left(\sum_{j=1}^{p} b_1^j\right)$ are even so we can call this element $2z$. Hence we achieved our first goal which was to find $2z \in \operatorname{Im} d^{1,p-2}$.

Note that since $C^3(G)$ doesn't have torsion, this implies that $z \in \ker d^{2,p-2}$.

$\triangle$ It remains to show that $z \notin \operatorname{Im} d^{1,p-2}$. Assume z can be written $z = d(x)$ for some $x \in C^{1,p-2}(G)$. We write the coordinates of x in the same basis of $C^{1,p-2}(G)$ as the one we previously used to write the matrix expression for $d^{1,p-2}$. Namely, the basis for $C^{1,p-2}(G)$ we use can be described as a partition $B_1 \sqcup B_2 \sqcup B_3$ where B_1 is the set of basis elements for which the present edge is e_1, B_2 is the set of basis elements b_i^j such that e_i is an edge in the cycle but is not e_1 and v_j is a vertex in the cycle, and B_3 is the set of all other basis elements b_i^j, i.e. either the ones for which e_i is an edge in the cycle but v_j is not a vertex in the cycle or e_i is not in the cycle. Using this basis, we can write x as a linear combination of basis elements, labeling its coordinates on elements of B_1 by α_i, its coordinates on elements of B_2 by β_i,and its coordinates on elements of B_3 by γ_i, as illustrated in the following matrix representation.

$$
\begin{array}{c|ccccc|c|c|c}
 & \multicolumn{5}{c|}{\overbrace{\qquad\qquad\qquad\qquad}^{\substack{e_1 \text{ and}\\ \text{another edge in cycle}}}} & \overbrace{\qquad\qquad}^{\substack{e_i,\, e_j \text{ in cycle,}\\ i,j\neq 1}} & \substack{\text{At least one edge}\\ \text{not in cycle}} & \\
\textit{Matrix of } d^{1,p-2} & a_{12} & a_{13} & \cdots & & a_{1,2s} & a_{23}\ \cdots\ a_{2s-1,2s} & \text{all other } a_{ij} & \text{Coeff} \\
\hline
 & -1 & & & & -1 & & & \alpha_1 \\
d(b_1^1) & -1 & -1 & & & & & & \alpha_3 \\
d(b_1^3) & & -1 & \ddots & & & 0 & M_1 & \vdots \\
 & & & \ddots & -1 & & & & \alpha_{2s-1} \\
d(b_1^{2s}) & & & & -1 & -1 & & & \alpha_{2s} \\
\hline
d(b_1^{2s+1}) & & & & & & & & \alpha_{2s+1} \\
\vdots & & & 0 & & & 0 & M_2 & \vdots \\
d(b_1^p) & & & & & & & & \alpha_p \\
\hline
d(b_i^j) & & & & & & & & \beta_1 \\
e_i,\, v_j \text{ in cycle} & & & M_3 & & & M_4 & M_5 & \vdots \\
i \neq 1 & & & & & & & & \beta_m \\
\hline
 & & & & & & & & \gamma_1 \\
\text{other } b_i^j & & & 0 & & & 0 & M_6 & \vdots \\
 & & & & & & & & \gamma_{m'} \\
\hline
d(x) = & -1 & -1 & \cdots & \cdots & -1 & 0\ \cdots\ 0 & -\delta_{ij} & = z
\end{array}
$$

Note that there is no b_1^2 in B_1 since b_1^2 and b_1^1 would be the same (and no corresponding α_2 coefficient). Hence the first block matrix with the -1's is a $(2s-1, 2s-1)$-matrix.

If we now read this by columns, what we get for the two first blocks of columns in the matrix representation, i.e. the columns corresponding to basis elements with both edges in the cycle, is the following:

$$
\begin{array}{rcccccc|cc}
\text{Column } 1(a_{12}): & -\alpha_1 & -\alpha_3 & & & & & +S_1 & = -1 \\
\text{Column } 2(a_{13}): & & -\alpha_3 & -\alpha_4 & & & & +S_3 & = -1 \\
\vdots & & & \ddots & \ddots & & & & \vdots \\
 & & & & -\alpha_{2s-1} & -\alpha_{2s} & & +S_{2s-2} & = -1 \\
\text{Column 2s-1}(a_{1,2s}): & -\alpha_1 & & & & -\alpha_{2s} & & +S_{2s-1} & = -1 \\
-------- & -- & -- & -- & -- & -- & - & -- & \\
\text{Column 2s}(a_{23}): & & & & & & & +S_{2s} & = 0 \\
\vdots & & & 0 & & & & & \vdots \\
 & & & & & & & & \\
\text{Column } \binom{2s}{2} (a_{2s-1,2s}): & & & & & & & +S_{\binom{2s}{2}} & = 0 \\
\hline
 & -2(\alpha_1 & +\alpha_3 & +\alpha_4 & \cdots & \cdots & +\alpha_{2s}) & +S_\beta & = 2s-1
\end{array}
$$

where the S_i's the result of the multiplication of M_3 and M_4 by the β_i's, read by columns. Their sum S_β is a linear combination of β_i's with coefficients in $\mathbb{Z}$. It suffices to prove that these coefficients are all even to get a contradiction, since this will imply that the left hand side is even while the right hand side is odd. This will be achieved by showing that there are exactly two non-zero entries in each row of the matrix $M = \begin{bmatrix} M_3 & | & M_4 \end{bmatrix}$, and that these entries are ± 1.

Indeed, for any b_i^j with e_i, v_j in the cycle, there are exactly two ways to add an edge adjacent to the component with the color 1 under the condition that this edge is in the cycle. The coordinates of $d(b_i^j)$ on these two basis elements is ± 1 and appear in M . The other basis elements in the image of b_i^j under the differential will be have at least one edge not in the cycle so the corresponding coordinate will appear in M_5.

□

4.2 What if we base the construction on other algebras?

In this case we know for sure that we can get torsions other than $\mathbb{Z}_2$.

Example 66. *Let $G = P_3$, the circuit graph with 3 vertices and 3 edges. Let $\mathcal{A} = \mathbb{Z}[X]/(X^3)$. Some computation, either using definition or using the exact sequence, shows*

$$H^0(P_3) \cong \mathbb{Z}\{3\} \oplus \mathbb{Z}^3\{4\} \oplus \mathbb{Z}^3\{5\} \oplus \mathbb{Z}\{6\}$$

and

$$H^1(P_3) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{2\} \oplus \mathbb{Z}_3\{3\}$$

We note that $H^{1,3}(P_3)$ has torsion of order 3. This is quite different from the known computations for knots, where it is conjectured no torsion of odd order can occur [S04].

More on torsion in this graph homology can be found in

Chapter 5

Questions: What's next

We ask a few questions that arise naturally in [HPR05]. our work.

Question 1. *What do these cohomology groups mean geometrically?*
The chromatic polynomial has a clear geometric interpretation. It is not clear what our cohomology groups measure.
We note that these cohomology groups are not functions of the matroid type of the graph since the graph made of two triangles glued at one vertex and the graph which is the disjoint union of two triangles have the same matroid type but different cohomology groups. In fact, they have different chromatic polynomials.

Question 2. *What is the relative strength of these invariants?*
▲ For any algebra $\mathcal{A}$ satisfying our assumptions, we now have several graph invariants that are by construction ordered by strength the following way:

Relative strength of our graph invariants for $\mathcal{A}$ *as in* (40) *with differential* $d = (m, id)$

$$
\begin{array}{ccc}
\text{isomorphism type of cochain complexes} & & \\
\downarrow (1) & & \\
\text{cohomology groups} & & \\
\downarrow (2) & & \\
\text{Poincaré polynomial } R_G(t,q) & & \text{Tutte polynomial + number of vertices} \\
\searrow (3) & & \swarrow (4) \\
& \text{chromatic polynomial} &
\end{array}
$$

The meaning of the arrows is the following: It is possible to recover the invariant at the head of the arrow starting from the one at the tail of the arrow.
Indeed, for a given graph G,
(1) from the cochain complex we can derive the cohomology groups.
(2) The Poincaré polynomial is a generating function that keeps track of the free rank of

the cohomology groups so we can derive it from the cohomology groups.
(3) Letting $t = -1$ in $R(t, q)$ yields the chromatic polynomial.
(4) The Tutte $T(G, x, y)$ and the chromatic polynomial $P(G, \lambda)$ of a graph G with $p(G)$ vertices and $k(G)$ connected components are related by the formula

$$P(G, \lambda) = (-1)^{p(G)-k(G)} \lambda^{k(G)} T(G, 1 - \lambda, 0).$$

Given that the degree of the chromatic polynomial is $p(G)$ and its leading coefficient is one, knowing the Tutte polynomial and the number of vertices of a graph is enough to recover the chromatic polynomial of a graph.
Note that knowing the Tutte polynomial and the number of connected components of a graph is also enough to recover the chromatic polynomial of a graph so we could have chosen that instead.

This raises a natural question: For a given algebra $\mathcal{A}$, if one of these invariants is potentially stronger than another one, is it actually stronger or are they equivalent? We expect the result to depend on the algebra that is being used.

We have the following partial results:

Relative strength of our graph invariants for $\mathcal{A}$ as in (40) with differential $d = (m, id)$

isomorphism type of cochain complexes

$\downarrow \not\uparrow$ (1)

cohomology groups

$\downarrow$ (2)

Poincaré polynomial $R_G(t, q)$ Tutte polynomial + number of vertices

$\searrow$ (3) $\swarrow \not\nearrow$ (4)

chromatic polynomial

(1) Isomorphism classes of cochain complexes are stronger than the cohomology groups since they can distinguish some graphs with loops whose cohomology groups are zero.
(4) All the graphs $L_n, n \geq 1$, have chromatic polynomial equal to zero. However, the Tutte polynomial of L_n is y^n.

However we do not know if there are graphs distinguishable by the cohomology groups but not by the chromatic polynomial (unless we allow generalized differentials, see example (59)).
Such examples would raise another, more subtle, question: Given two graphs with the same chromatic polynomial can one always find an algebra $\mathcal{A}$ such that the homolog groups over $\mathcal{A}$ distinguish these two graphs?

We do know that the Tutte polynomial can sometimes distinguish graphs with the same cohomology groups (e.g. graphs with loops all have 0 cohomology groups). But we don't know whether there are examples in the other direction.

▲ If we allow a larger class of differentials, as explained in (43), we may get different results. For instance recall that in example (59),where $\mathcal{A} = \mathbb{Z}X$, with $X^2 = 0$, and the differential is zero, the cochain complex and the cohomology groups contain exactly the same information since the cohomology groups are equal to the cochain groups.

Question 3. *What is the relationship with the Khovanov cohomology for knots?*
We expect such a relationship given the known relationship between Jones type knot invariants and Tutte type invariants for graphs. For example, Theorem (28) on pendant edges should corresponds to the change of Khovanov cohomology for framed links under type 1 Reidemeister move. Note, however, that the Jones polynomial does not correspond to the chromatic polynomial, instead, it corresponds to a specialization of the Tutte polynomial. Thus some work is still needed to uncover the expected relationship.

In [HPR05], we explain some relations between the graph cohomology of a planar graph and the Khovanov cohomology of its Tait link. The version of Khovanov cohomology used there is the one for framed unoriented links defined by Viro in Section 6 of [V04].

Question 4. *What kind of torsion can these cohomology groups contain?*
We expect the result to depend on the algebra that is being used. For instance, when the algebra is $\mathcal{A} = \mathbb{Z}[X]/(X^2)$ the torsion elements in our examples are all of order 2. We would like to know whether the order of torsion elements are all 2, or perhaps powers of 2, in general when the algebra is $\mathcal{A} = \mathbb{Z}[X]/(X^2)$.
We already know under which circumstances these groups have torsion when the algebra is $\mathcal{A} = \mathbb{Z}[X]/(X^2)$: A loopless graph will always have a $\mathbb{Z}_2$ torsion provided it contains a cycle of length at least 3.
Again, we expect the result to depend on the algebra. For instance, when the algebra is $\mathbb{Z}[X]/(X^3)$, we already observed in example (55) that we can get a $\mathbb{Z}_3$ torsion.

Some results in this direction have been obtained in [HPR05]. For instance, it is shown in that paper that $H^{1,m}_{\mathcal{A}_m}(P_3) = \mathbb{Z}_m$ where $\mathcal{A}_m$ is the algebra $\mathbb{Z}[X]/(X^m)$. This shows that any torsion can occur.

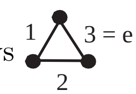
Question 5. *Can a similar construction be made for the Tutte polynomial?*
The Tutte polynomial can be expressed as a state sum so we have something to start from but the fact that it is a two-variable polynomial makes things more difficult. The standard techniques of categorification work for one-variable polynomials. The categorifications of two variable polynomials that have been done so far, i.e. the categorification of the HOMFLY polynomial [KR04] and of the Dichromatic and the Tutte polynomial for

graphs [St05] are indeed categorifications of a sequence of one-variable specializations of the polynomial that are enough to recover the original polynomial. So there is still the need for a categorification of the Tutte polynomial as a two variable polynomial.

Chapter 6

Appendices: Some computational examples

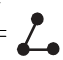
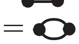
6.1 Computation of the cohomology groups of the graph P_3 (the triangle) via the exact sequence when $\mathcal{A} = \mathbb{Z}[X]/(X^2)$

Let $G = P_3$. We label the edges the following ways , which respects the rule that when we use the exact sequence the edge e has to be the last one in the ordering.

The exact sequence with respect to (G, e) is:

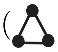
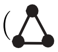
$0 \to H^0($$) \to H^0($$) \xrightarrow{\gamma^*} H^0($$) \to H^1($$) \to H^1($$) \to \cdots$

▲ **Step 1: Reduce the problem.**

△ *Substitute in the exact sequence the cohomology groups that are known.*The cohomology groups are known for $G - e$ and G/e: Indeed, $G - e =$  is a tree with 2 edges and since the cohomology groups don't see multiple edges, $G/e =$  has the same cohomology groups as , which is also a tree. In both cases, the cohomology groups can be obtained from the the cohomology groups of the graph made of an isolated vertex by raising the degrees by one for each edge in the tree as explained in Theorem (28), the pendant edge theorem. One can also use directly the result provided in Example (33), the example of a tree with n edges.

△ *Split the exact sequence by degree.* Since all the groups in the exact sequence are graded and all the maps in the exact sequence are degree preserving, we can split the exact sequence by degree. This is what the row labels "$j = \cdots$" mean. For instance, the row "$j = 2$" deals with the elements of degree 2. In this same row, γ_{02}^* is the restriction of γ_0^* to the elements of degree 2.

The exact sequence becomes:

	$0 \to$	$H^0($$)$	$\to$	$H^0($$)$	$\xrightarrow{\gamma^*}$	$H^0($$)$	$\to$	$H^1($$)$	$\to$	$H^1($$)$	$\to \cdots$
		G		$G-e$		G/e		G		$G-e$	
$j=0$	$0 \to$	$H^{00}(G)$	$\to$	0	$\to$	0	$\to$	$H^{10}(G)$	$\to$	0	$\to \cdots$
$j=1$	$0 \to$	$H^{01}(G)$	$\to$	0	$\to$	$\mathbb{Z}$	$\to$	$H^{11}(G)$	$\to$	0	$\to \cdots$
$j=2$	$0 \to$	$H^{02}(G)$	$\to$	$\mathbb{Z}$	$\xrightarrow{\gamma^*_{02}}$	$\mathbb{Z}$	$\to$	$H^{12}(G)$	$\to$	0	$\to \cdots$
$j=3$	$0 \to$	$H^{03}(G)$	$\to$	$\mathbb{Z}$	$\to$	0	$\to$	$H^{13}(G)$	$\to$	0	$\to \cdots$

△ *Use the vanishing theorem and the thickness theorem to determine which groups we need to compute.*

Note that a reader not acquainted yet with these theorem may safely skip this paragraph since the result we get here can also be obtained using Step 2 of the method.

⋄ Theorem (36), the vanishing theorem, says that for $i \neq 0$, $H^i(G) = 0$ unless maybe if $i \leqslant p-2$. This shows that we only have to take care of $H^0(G)$ and $H^1(G)$.

⋄ Theorem (37), the thickness theorem, says that if G is a connected graph with p vertices, then $H^{i,j}(G) = 0$ unless maybe if $p-1 \leqslant i+j \leqslant p$. In our case, it means that $H^{i,j}(G) = 0$ unless maybe if $2 \leqslant i+j \leqslant 3$.

We get that $H^{00}(G)$, $H^{01}(G)$, $H^{10}(G)$ and $H^{13}(G)$ are trivial.

▲ Step 2: Some cohomology groups are now obvious

We now use the two following basic properties of exact sequences to derive some cohomology groups.

Rule 1: If $0 \to G \to 0$ is exact then $G \cong 0$.

Rule 2: If $0 \to G_1 \to G_2 \to 0$ is exact then $G_1 \cong G_2$.

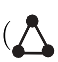
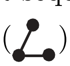
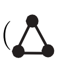
By Rule 2, we get that $H^{03}(G) \cong H^{11}(G) \cong \mathbb{Z}$.

In this specific example, the vanishing and the thickness theorem were not needed since Rule 1 also proves that $H^{00}(G)$, $H^{01}(G)$, $H^{10}(G)$ and $H^{13}(G)$ are trivial.

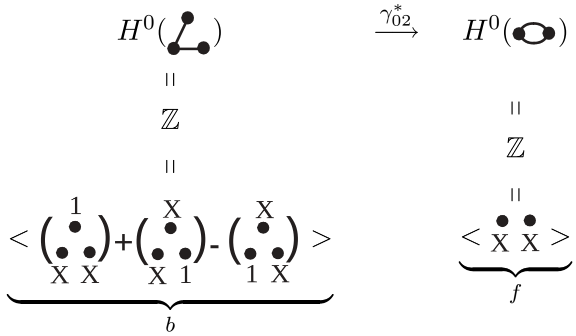
▲ Step 3: Remaining cohomology groups

The only cohomology groups not determined so far are $H^{02}(G)$ and $H^{12}(G)$.

If we restrict the exact sequence to the elements of degree 2, we have:

$0 \to H^{02}($$) \xrightarrow{\beta^*_{02}} H^{02}($$) \xrightarrow{\gamma^*_{02}} H^{02}($$) \xrightarrow{\alpha^*_{02}} H^{12}($$) \xrightarrow{\beta^*_{12}} H^{12}($$) \to \cdots$

We want to determine γ^*_{02}



Generators for these groups can be obtained either via the pendant edge algorithm (Proposition (30)), or using the description of basis elements of cohomology groups for trees in section (2.6) or using the method described in the proof of Theorem (38), the 0-cohomology theorem.

Using the definition of γ^* described in remark(21), we get $\gamma_{02}^*(b) = 2f$.

This implies that γ_{02}^* is injective so $ker\gamma_{02}^* = 0$ and that $Im\gamma_{02}^* =< 2f >$. We combine these facts with the information of what the exact sequence is:

$0 \to H^{02}(G) \overset{\beta_{02}^*}{\to} < e > \overset{\gamma_{02}^*}{\to} < f > \overset{\alpha_{02}^*}{\to} H^{12}(G) \to 0$

$\diamond$ β_{02}^* is injective so $H^{02}(G) \cong Im\beta_{02}^* = ker\gamma_{02}^* = 0$

$\diamond$ α_{02}^* is surjective so by the first isomorphism theorem,

$H^{12}(G) = Im\alpha_{02}^* \cong < f > /ker\alpha_{02}^* =< f > /Im\gamma_{02}^* =< f > / < 2f > \cong \mathbb{Z}_2$.

Summary : $H^0_{\mathbb{Z}[X]/(X^2)}(\;) = \mathbb{Z}\{3\}$, as predicted by the 0-cohomology theorem. $H^1_{\mathbb{Z}[X]/(X^2)}(\;) = \mathbb{Z}\{1\} \oplus \mathbb{Z}_2\{2\}$, and $H^i_{\mathbb{Z}[X]/(X^2)}(\;) = 0$ if $i \geq 2$.

6.2 Classification of rings of the form $\mathcal{A} = \mathbb{Z}1 \oplus \mathbb{Z}x$ with 1 and x of degree 0

Here, we would like to classify all commutative rings with identity whose additive group is free abelian group of rank two. Let R be such a ring. Its additive group $(R, +)$ is generated by 1 and x. We have $1 * 1 = 1, 1 * x = x * 1 = x$, and $x * x = a1 + bx$ where a, b are arbitrary integers. Obviously the ring structure of R is completely determined by (a, b). Let R' be another ring whose additive group is generated by $1'$ and x' with $x' * x' = a'1' + b'x'$. We have

Proposition 67. *The two rings are isomorphic if and only if $b^2 + 4a = b'^2 + 4a'$ and $b \equiv b'$ (mod 2)). In other words, the isomorphism type of R is completely determined by $(b^2 + 4a$, $b \pmod 2) \in (\mathbb{Z}, \mathbb{Z}_2)$.*

Proof. Suppose $R \cong R'$ as modules, and let $f : R \to R'$ be an isomorphism. Then $f(1) = 1', f(x) = k1' + lx'$ where k, l are integers. Since $\{f(1), f(x)\}$ spans the R' as an abelian group, we have $l = \pm 1$.

Since f preserves the multiplication, $f(x^2) = (k1' + lx')^2 = k^2 + 2klx' + l^2x'^2 = k^2 + 2klx' + l^2(a'1' + b'x') = (k^2 + l^2a')1' + (2kl + l^2b')x'$. We also have $f(a1 + bx) = a1' + b(k1' + lx') = (a + kb)1' + lbx'$. Since $x^2 = a1 + bx$, we have

$$\begin{cases} k^2 + l^2a' = a + kb & (1) \\ 2kl + l^2b' = lb & (2) \end{cases}$$

The above two equations are equivalent to the following, where (3) is obtained by taking (1)$-\frac{k}{l}*$(2), and (4) is $\frac{1}{l}*$(2):

$$\begin{cases} l^2a' - klb' - k^2 = a & (3) \\ 2k + lb' = b & (4) \end{cases}$$

Take $4*(3)+(4)^2$, we obtain:

$$l^2(b'^2 + 4a') = b^2 + 4a$$

This is equivalent to $b'^2 + 4a' = b^2 + 4a$ since $l = \pm 1$.

The relation $b \equiv b' \pmod 2$ follows from equation (4) by taking mod 2 in both sides.

Conversely, if (a, b) and (a', b') satisfy the two relations, we let $l = 1$, and $k = \frac{1}{2}(b - b')$. A straight forward computation shows that equations (1) and (2) hold. This implies that the map $f : R \to R'$ defined by $f(1) = 1', f(x) = k1' + x'$ is a ring isomorphism.

6.3 A few computational results

In the array that keeps track of the cohomology groups, the numbers without brackets indicate the number of copies of $\mathbb{Z}$ while the numbers with brackets indicate the number of copies of $\mathbb{Z}_2$.

For instance, in the case of the triangle (see figure (6.1) below), the first column means $H^0(P_3) = \mathbb{Z}\{3\}$ ($= H^{03}(P_3)$, elements of degree 3). The second column means $H^{11}(P_3) = \mathbb{Z}\{1\}$ (elements of degree 1 in H^1) and $H^{12}(P_3) = \mathbb{Z}_2\{2\}$ (elements of degree 2 in H^1) i.e. $H^1(P_3) = \mathbb{Z}\{1\} \oplus \mathbb{Z}_2\{2\}$

The names G_{40} and G_{42} of the graphs in the Figure (6.5) follow the classification in [RW99].

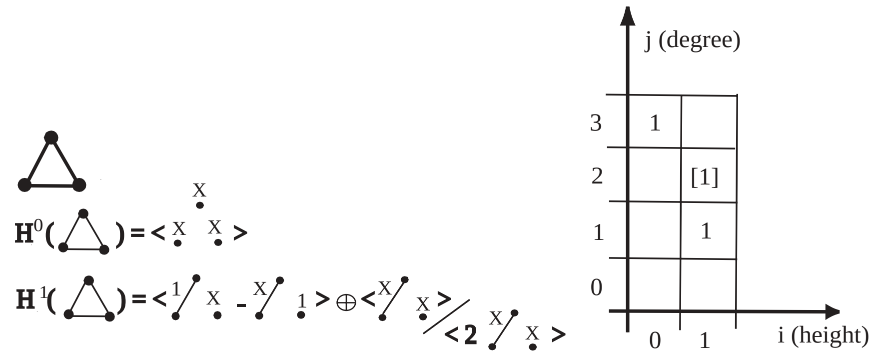


Figure 6.1: P_3 *summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

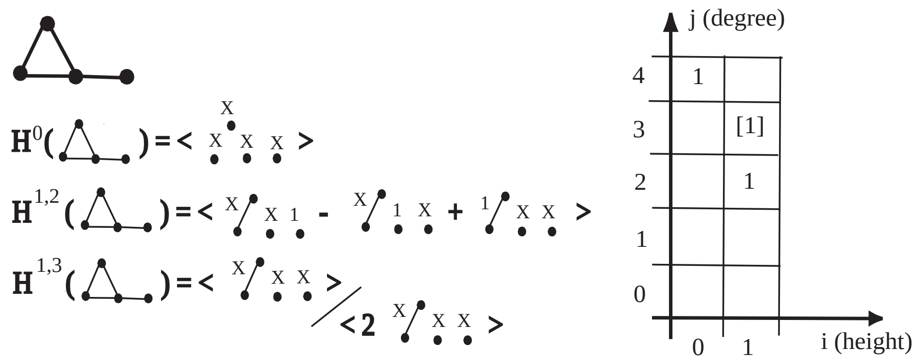


Figure 6.2: P_3 *with a pendant edge, summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

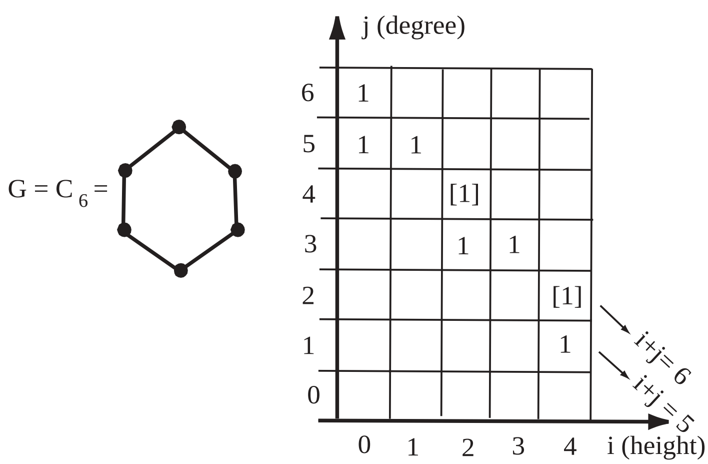


Figure 6.3: P_6 *summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

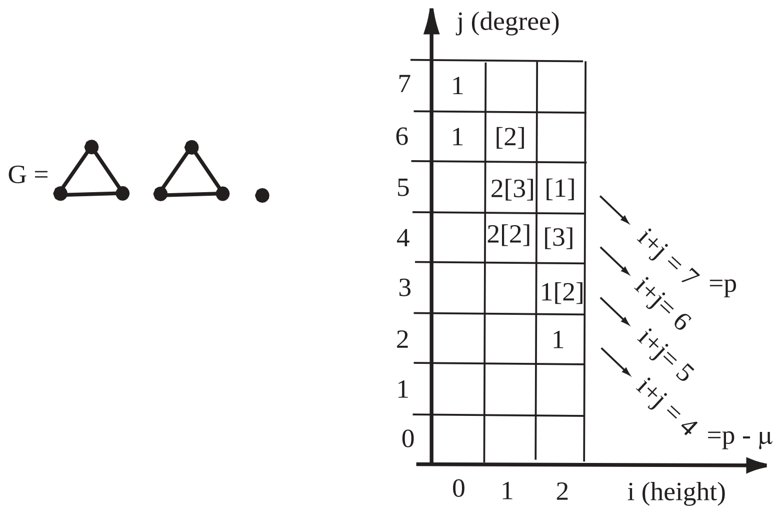


Figure 6.4: *Two triangles and an isolated vertex, summary when the algebra is* $\mathbb{Z}[X]/(X^2)$

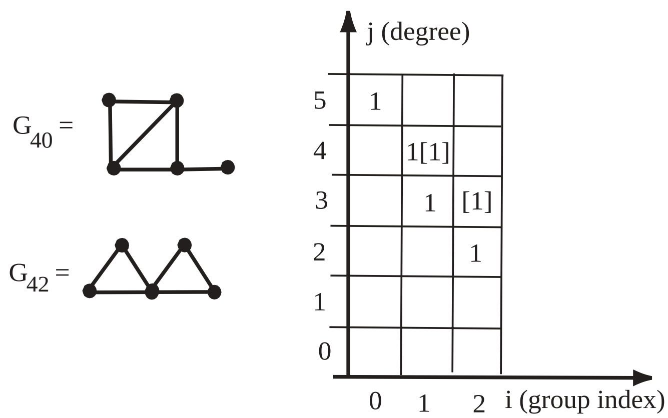


Figure 6.5: *summary for* G_{40} *and* G_{42} *when the algebra is* $\mathbb{Z}[X]/(X^2)$

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