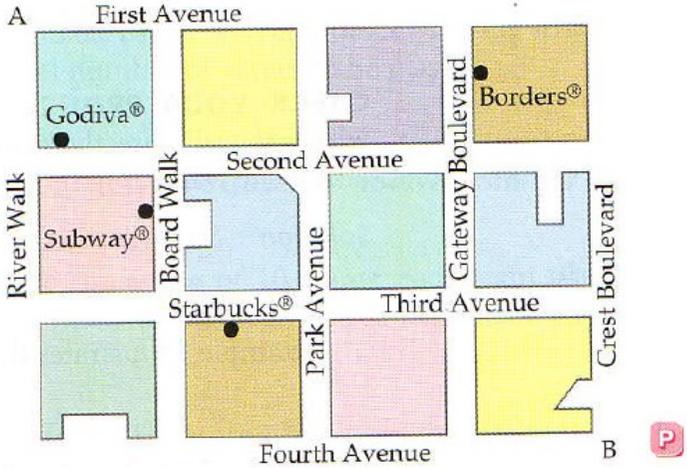


I-Section Européenne DNL mathématiques <b>Problem solving with Polya's strategy :</b> <b>Number of direct routes</b>	2 <sup>nde</sup> 2012-13
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Source : *Mathematical excursions* by Aufman, Lockwood, Nation and Clegg.

Consider the map shown below. Allison wishes to walk along the streets from point A to point B. How many direct routes (no backtracking) can Allison take?



## Section Européenne DNL mathématiques

# Polya's problem solving strategy

Source : *Mathematical excursions* by Aufman, Lockwood, Nation and Clegg.

### historical note



#### George Polya

After a brief stay at Brown University, George Polya (pōl'ya) moved to Stanford University in 1942 and

taught there until his retirement. While at Stanford, he published 10 books and a number of articles for mathematics journals. Of the books Polya published, *How to Solve it* (1945) is one of his best known. In this book, Polya outlines a strategy for solving problems from virtually any discipline.

"A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery." ■

### Polya's Problem-Solving Strategy

Ancient mathematicians such as Euclid and Pappus were interested in solving mathematical problems, but they were also interested in *heuristics*, the study of the methods and rules of discovery and invention. In the seventeenth century, the mathematician and philosopher René Descartes (1596–1650) contributed to the field of heuristics. He tried to develop a universal problem-solving method. Although he did not achieve this goal, he did publish some of his ideas in *Rules for the Direction of the Mind* and his better-known work *Discourse de la Methode*.

Another mathematician and philosopher, Gottfried Wilhelm Leibnitz (1646–1716), planned to write a book on heuristics titled *Art of Invention*. Of the problem-solving process, Leibnitz wrote, "Nothing is more important than to see the sources of invention which are, in my opinion, more interesting than the inventions themselves."

One of the foremost recent mathematicians to make a study of problem solving was George Polya (1887–1985). He was born in Hungary and moved to the United States in 1940. The basic problem-solving strategy that Polya advocated consisted of the following four steps.

#### Polya's Four-Step Problem-Solving Strategy

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Review the solution.

Polya's four steps are deceptively simple. To become a good problem solver, it helps to examine each of these steps and determine what is involved.

**Understand the Problem** This part of Polya's four-step strategy is often overlooked. You must have a clear understanding of the problem. To help you focus on understanding the problem, consider the following questions.

- Can you restate the problem in your own words?
- Can you determine what is known about these types of problems?
- Is there missing information that, if known, would allow you to solve the problem?
- Is there extraneous information that is not needed to solve the problem?
- What is the goal?

**Devise a Plan** Successful problem solvers use a variety of techniques when they attempt to solve a problem. Here are some frequently-used procedures.

- Make a list of the known information.
- Make a list of information that is needed.
- Draw a diagram.
- Make an organized list that shows all the possibilities.
- Make a table or a chart.
- Work backwards.
- Try to solve a similar but simpler problem.
- Look for a pattern.
- Write an equation. If necessary, define what each variable represents.
- Perform an experiment.
- Guess at a solution and then check your result.
- Use indirect reasoning.

**Carry Out the Plan** Once you have devised a plan, you must carry it out.

- Work carefully.
- Keep an accurate and neat record of all your attempts.
- Realize that some of your initial plans will not work and that you may have to devise another plan or modify your existing plan.

**Review the Solution** Once you have found a solution, check the solution.

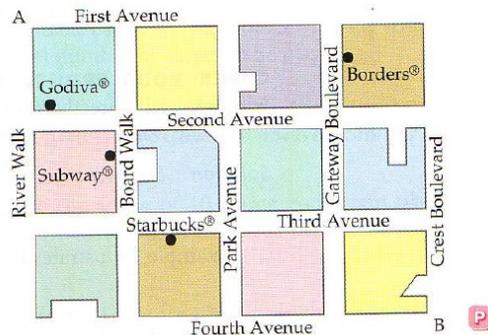
- Ensure that the solution is consistent with the facts of the problem.
- Interpret the solution in the context of the problem.
- Ask yourself whether there are generalizations of the solution that could apply to other problems.

In Example 1 we apply Polya's four-step problem-solving strategy to solve a problem involving the number of routes between two points.

**Section Européenne DNL mathématiques**  
**Apply Polya's strategy:**  
**Solve a similar but simpler problem**  
**Number of direct routes**

Source : *Mathematical excursions* by Aufman, Lockwood, Nation and Clegg.

Consider the map shown below. Allison wishes to walk along the streets from point A to point B. How many direct routes can Allison take?

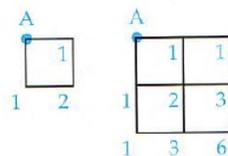


**Solution**

**Understand the Problem** We would not be able to answer the question if Allison retraced her path or traveled away from point B. Thus we assume that on a direct route, she always travels along a street in a direction that gets her closer to point B.

**Devise a Plan** The map in Figure 1.2 has many extraneous details. Thus we make a diagram that allows us to concentrate on the essential information. See the figure at the left.

Because there are many routes, we consider the similar but simpler diagrams shown below. The number at each street intersection represents the number of routes from point A to that particular intersection.



Look for patterns. It appears that the number of routes to an intersection is the *sum* of the number of routes to the adjacent intersection to its left and the number of routes to the intersection directly above. For instance, the number of routes to the intersection labeled 6 is the sum of the number of routes to the intersection to its left, which is three, and the number of routes to the intersection directly above, which is also three.

**Carry Out the Plan** Using the pattern discovered on the previous page, we see from the figure at the left that the number of routes from point A to point B is  $20 + 15 = 35$ .

**Review the Solution** Ask yourself whether a result of 35 seems reasonable. If you were required to draw each route, could you devise a scheme that would enable you to draw each route without missing a route or duplicating a route?

**CHECK YOUR PROGRESS 1** Consider the street map in Figure 1.2. Allison wishes to walk directly from point A to point B. How many different routes can she take if she wants to go past Starbucks on Third Avenue?

Section Européenne DNL mathématiques, Seconde 2010-2011

Problem solving with Polya's strategy

You've got Problems!

Group work : Oral presentations + written report

■ Directions

Work in groups of four students. One of the problems below will be assigned to your group. You will have to present the problem and its solution to the class and also to turn in a written report about your findings. Make sure you split the oral presentation (so each member of your group gets a chance to talk) and the writing of the report (everyone's handwriting should appear in the written report)<sup>1</sup>.

■ Oral presentation: On Monday October 8<sup>th</sup>.

- Describe the problem to the class. If you wish, while a person talks, another member of the team can draw a diagram on the board to explain the situation or the strategy. You can also bring a poster you made in advance and hang it or you can prepare a PowerPoint presentation.
- Explain which difficulties you ran into while working on the problem. Explain your strategy. If you had several ideas, explain why you picked this one. Explain the steps you took, focusing on the ideas rather than on the computations. The details of the computations should go in the written report.
- It should be clear that you applied the four steps of Polya's problem solving strategy.
- Be ready to answer questions from your classmates at the end of your presentation (which in turn implies that you should also be ready to ask questions about the work of the other teams!)

■ Written report: due on Monday October 15<sup>th</sup>.

- Explain your strategy and the steps you took.
- *Stretch it (extra credit question)*: If you have some time left, take the problem one step further: Your problem may naturally lead to another question (getting a more precise answer, or stating and proving a generalization of your result). State the question and answer it.

■ <u>How your work will be assessed:</u>	Student	Team
<b>In class group work:</b>		
Used English only while working on the project in class.		X /3
Worked cooperatively with other team members: listened to other team members, offered ideas. Made constructive rather than negative comments on the work of other team members.		X /3
Worked quietly so as not to disturb the other teams (please whisper!)		X /2
<i>Subtotal</i>		/8
<b>Oral presentation (each member of the team should talk in turn):</b>		
The problem was explained clearly.	X	/3
The strategy (and why you chose it) was explained clearly.	X	/3
The steps taken were explained clearly. The four steps of Polya's problem solving strategy were used.	X	/4
English didn't get in the way. The reporter was talking rather than reading his/her notes.		X /5
<i>Subtotal</i>		/15
<b>Written report (each member of the team should write a part of it):</b>		
The solution is correct in terms of mathematics and logic.	X	/5
The English is correct.	X	/2
The presentation is well organized and provides the necessary details	X	/3
<i>Bonus: Stretch it.</i>	X	/4
<i>Subtotal</i>		/10
<b>Class sharing</b>		
Listened carefully to the others teams' presentations.		X /2
<i>Bonus: Offered good comments, asked relevant questions during the others teams' presentations.</i>		X /2
<i>Subtotal</i>		/2
GRADE		X /35
GRADE		X /20

<sup>1</sup> If you do not talk during the oral presentation, you will be assigned a grade of 0 for this part of the project. Similarly, if you handwriting does not appear in the written report, you will be assigned a grade of 0 for this part of the project.

■ Problem 1.

Determine the digit in the 100 place to the right of the decimal point in the decimal representation of  $\frac{7}{27}$ .

■ Problem 2.

A traffic counter is a device designed to count the number of vehicles passing along a street. It usually is a thin black tube stretched across a street or highway, and connected to a "brain box" at the side of the road. The device registers one "count" each time a set of wheels (that is, wheels on a single axis) rolls over the tube. A normal automobile registers two counts: one for the front wheels and one for the rear wheels. A light truck with three axles registers three counts. A large semitrailer truck might register four or five counts.

Suppose that, during a one-hour period, a particular traffic counter registers 35 counts on a residential street on which only two axles (cars) and three axles (light trucks) are allowed. How many cars and light trucks passed over the traffic counter?

■ Problem 3. On your marks, Get set, Go!

Jill and Jack ran a 100-meters race. Jill won by 5 meters; that is Jack had run only 95 meters when Jill crossed the finish line. They decide to race again but this time, Jill starts 5 meters behind the starting line. Assuming that both runners run at the same pace as before, who will win?

■ Problem 4.

The product of ages of three teenagers is 4590. None of the teens are the same age. What are the ages of the teenagers?

■ Problem 5.

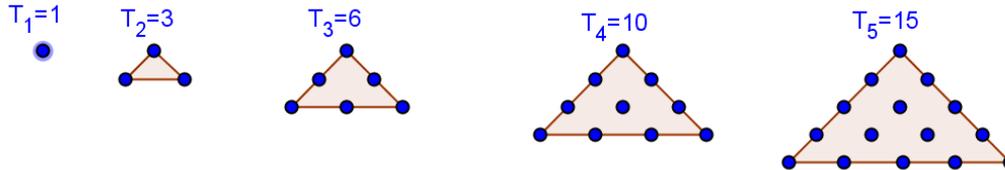
Determine the ones digit of  $4^{200}$ .

■ Problem 6.

In a basketball league consisting of 10 teams, each team must play each of the other teams exactly three times. How many league games will be played?

■ Problem 7. Triangular numbers

The ancient Greek mathematicians were interested in the geometric shape associated with numbers. For instance, they noticed that triangles can be constructed using 1, 3, 5, 10 and 15 dots as shown on the figure below.



Therefore, they called the numbers 1, 3, 5, 10, 15, ... the triangular numbers.

Determine  $T_6$ , the 6<sup>th</sup> triangular number, and then find a general formula for the  $n^{\text{th}}$  triangular number.

■ Problem 8.

$2!$  (read "2 factorial") is defined by  $2! = 2 \cdot 1$ ,  $3! = 3 \cdot 2 \cdot 1$ ,  $4! = 4 \cdot 3 \cdot 2 \cdot 1 \dots$  etc.

- 1) Determine the last two digits of  $17!$
- 2) How many times does last two digit of  $627!$  repeat at the end of this number?

■ Problem 9. Coiled wire

Let's consider a problem of a type often encountered by plumbers, electricians and engineers: measuring or wrapping a wire around a cylindrical pipe. Suppose that eight turns of a wire are wrapped around a pipe with length of 20 centimeters and a circumference of 6 centimeters. What is the length of the wire?

■ Problem 10.

In a football tournament every team plays a match against every other team once and only once. If 120 matches are played in total, how many teams participated?

■ Problem 11.

Suppose you have two piles of marbles in front of you: One pile of white marbles and one pile of black marbles. Each pile initially has 10 marbles. Suppose that on the first transfer, three black marbles are moved to the white pile. On the second transfer, any three marbles are taken from the white pile and put into the black pile. Will you always end up with as many white marbles in the black pile as black marbles in the white pile?

■ Problem 12. The cars and the Canary

Two cars, 120 miles apart, begin driving toward each other on a long straight highway. One car travels 20 miles per hour and the other 40 miles per hour. At the same time a canary, starting on one car, flies back and forth between the two cars as they approach each other. If the canary flies always 150 miles per hour and turns around instantly at each car, how far has it flown when the cars collide?

■ Problem 13.

How many ways can a person climb a set of stairs with  $N$  steps given that he/she can climb one step at a time or skip a step?

Start with  $N=2$ , then  $N=3$  before you attempt to solve the general case.

■ Problem 14.

A lady always arrives at the train station at exactly six o'clock to pick up her husband and drive him home. One day the man arrives an hour early, starts walking home, and is picked up by the lady on his way out to the train station. They arrive at home twenty minutes earlier than usual. How long did he walk before he met his wife?

Teacher's worksheet  
 Section Européenne DNL mathématiques, Seconde 2010-2011  
 Problem solving with Polya's strategy  
 You've got Problems!

Group work : Oral presentations + written report

<b>■ <u>How work will be assessed:</u></b>	Students					Team	
<b>In class group work:</b>							
Used English only while working on the project in class.						X	/3
Worked cooperatively with other team members: listened to other team members, offered ideas. Made constructive rather than negative comments on the work of other team members.						X	/3
Worked quietly so as not to disturb the other teams (please whisper!)						X	/2
<b>Subtotal</b>							<b>/8</b>
<b>Oral presentation (each member of the team should talk in turn):</b>							
The problem was explained clearly.	X	X	X	X	X		/3
The strategy (and why you chose it) was explained clearly.	X	X	X	X	X		/3
The steps taken were explained clearly. The four steps of Polya's problem solving strategy were used.	X	X	X	X	X		/4
English didn't get in the way. The reporter was talking rather than reading his/her notes.						X	/5
<b>Subtotal</b>							<b>/15</b>
<b>Written report (each member of the team should write a part of it):</b>							
The solution is correct in terms of mathematics and logic.	X	X	X	X	X		/5
The English is correct.	X	X	X	X	X		/2
The presentation is well organized and provides the necessary details	X	X	X	X	X		/3
<i>Bonus: Stretch it.</i>	X	X	X	X	X		/4
<b>Subtotal</b>							<b>/10</b>
<b>Class sharing</b>							
Listened carefully to the others teams' presentations.						X	/2
<i>Bonus: Offered good comments, asked relevant questions during the others teams' presentations.</i>						X	/2
<b>Subtotal</b>							<b>/2</b>
GRADE						X	/35
<b>GRADE</b>						X	<b>/20</b>

Section Européenne DNL mathématiques

**Apply Polya's strategy:**

**Solve a similar but simpler problem**

**Number of direct routes**

Classe : Seconde

Objectifs :

- Leur donner un problème où *tout le monde* peut démarrer (on peut expérimenter)
- Découvrir une stratégie de résolution de problème : Schématiser la situation, expérimenter, démarrer par un cas simple que l'on généralise : L'art de ne pas rester paralysé devant un problème.
- Appliquer la méthode de Polya de résolution de problème.
- Assimilation de la méthode par une démarche active : « Check your progress » permet de réinvestir immédiatement.

Déroulement prévu pour la séance :

- Leur donner le problème et les laisser chercher seuls.
- Après quelques minutes, expliquez la question si besoin est.
- Leur distribuer le document « Polya's method » et de nouveau, les laisser chercher. (On peut éventuellement les mettre en groupe et demander un rapporteur par groupe ?)
- Leur distribuer le corrigé et leur demander de faire le « Check your progress »

Prolongement :

- Leur donner des problèmes à résoudre par groupe (tirage au sort) et à présenter à l'oral (liste de problèmes ci-dessus).

Réaction des élèves : Testé deux années de suite, cela fonctionne bien malgré des problèmes de réactions agressives de certains élèves aux questions jugées bêtes de la salle la deuxième fois. Ils découvrent la difficulté de se faire comprendre quand on parle d'un sujet sur lequel on a réfléchi à quelqu'un qui n'y a pas réfléchi du tout. Peut-être verront-ils les profs avec plus de sympathie ? Cela a aussi mis en évidence qu'ils ne savent pas lire les réactions de l'auditoire et peuvent donc noyer tout le monde sans réaliser qu'il y a un problème.

### Presentation pour le site Euromath

Une activité avec une première phase de réflexion sur les techniques de résolution de problèmes à partir d'un exemple puis une seconde phase où les élèves travaillent par groupe de quatre pour résoudre des problèmes de mathématiques (un peu) ouverts. La grille de notation qui leur est donnée dès le début du projet leur permet de savoir quel sont les attendus, notamment qu'ils doivent communiquer entre eux en anglais uniquement sinon ils perdent des points. Ils font une présentation orale (du type narration de recherche) puis un compte-rendu écrit à rendre une semaine plus tard et qui peut incorporer de nouvelles idées suites aux commentaires de la salle lors de la présentation orale.

Chers collègues : Pour me faire parvenir vos commentaires sur ce document et/ou échanger des idées sur la DNL, vous pouvez me contacter à [lhelmeg@yahoo.com](mailto:lhelmeg@yahoo.com)

Problèmes en gestation :

#### **■ Problem**

Diophantus's age, p 36. *Mathematical excursions* by Aufman, Lockwood, Nation and Clegg.