Section Européenne DNL mathématiques **(Basic) arithmetics**	1ère 2013-14

Part II is based on a document by the Académie de Bordeaux.

I. Multiples and Divisors

Definition 1. Let n and m be integers. m is a multiple of n if and only if there exists an integer k such that $n=k\,n$. In this case, we also say that n is a divisor of m.

For example, 63 is a multiple of 9 because $63=9\times7$. $(m=63;n=9;k=7)$

Problem 1.

1) List all the divisors of 28. Add all of them but 28.

→ *In number theory, a perfect number is a natural number that is equal to the sum of its positive divisors excluding the number itself.*

2) There exists only one perfect number which is less than 10. Can you find it?

Problem 2.

A long time ago you learnt that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. Prove it in the case of a three digit number.

II. Prime numbers

A. Understanding what a prime number is

Definition 2. A ***prime number*** (or a ***prime***) is a natural number greater than 1 that has no positive divisors other than 1 and itself. A natural number greater than 1 that is not a prime number is called a ***composite number***.

For example, 5 is prime because only 1 and 5 evenly divide it, whereas 6 is composite because it has the divisors 2 and 3 in addition to 1 and 6.

Useful Words

- prime /praɪm/
- divisor /dɪˈvaɪ zər/
- sieve /sɪv/ =*crible*
- composite /kəmˈpɒz ɪt/

Note : 1 is NOT prime.

Property 3. Every natural number (equal to or greater than 2) has got at least one prime divisor.

Problem 3. Find the lowest prime divisor of the following numbers : 12 13 187 17^2

B. Tests of primality: They tell you whether a given number is prime

Problem 4. ***Primality (first approach)***

1) Is 4347 a prime number? *Justify your answer.*

2) Is 43 a prime number? *Describe your method.*

Problem 5. ***Eratosthenes' sieve /ˌɛrəˈtɒsθəˌniz sɪv/***

The aim of the exercise is to present and to apply a method to find all the prime numbers lower than 100 without any calculator.

Here's a table of the natural numbers which are lower than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1) We want to keep the prime numbers, and to eliminate the numbers which are not prime. What's the first eliminated number?...

2) a) Is two a prime number? …………………………………………………………… ….
b) What are the multiples of two? ……………………………………………………… …….
c) Why must they be crossed out? ……………………………………………………… …………

3) Repeat the operation with the next prime numbers: 3 and its multiples, 5 and 7.

If we go on, the last prime number used in the algorithm is 11. Its multiples have already been eliminated because 11×2, 11×3, 11×5… have already been crossed out as multiples of 2, 3, 5…and $11 \times 11 > 100$!
It will be the same for the following numbers: 13×2, 13×3, 13×5… have already been crossed out! So all the remaining numbers are prime and we can stop there the algorithm.

→ ***This algorithm is called the "Sieve of Eratosthenes" [Crible d'Ératosthène].***

Problem 6. Eratosthenes
1) When and where did Eratosthenes live?
2) Which discipline did he create?
3) What are is main achievements?

Once you are done, you may take a look at : http://en.wikipedia.org/wiki/Sieve_of_Eratosthenes to see the sieve of Eratosthenes in action!

III. The Fundamental theorem of Arithmetic

What is it? http://www.mathsisfun.com/numbers/fundamental-theorem-arithmetic.html

Problem 7. How to use a tree to find prime factors.
http://www.bbc.co.uk/schools/gcsebitesize/maths/number/primefactorshirev1.shtml

IV. Highest Common Factor (HCF) and Lowest Common Multiple (LCM)

Problem 8. http://www.bbc.co.uk/schools/gcsebitesize/maths/number/primefactorshirev1.shtml

Problem 9. Capucine and Kelly go shopping for new mobile phones. Capucine gets a contract that lasts 25 months. Kelly gets a 10 month contract. They want to keep renewing their contracts until they can shop for mobiles together again. How long will this take?

Problem 10. Fun with HCF, LCM a pentagon and and on hexagon. See below.

Problem 11. Anna wants to buy her sisters Raphaelle and Celia some sweets. The sweets Celia likes come in bags of 18, but Raphaelle's favourite sweets come in bags of 27. Anna wants to buy an equal amount of sweets for both her sisters, so has to buy different quantities of each packet. How many sweets will Raphaelle and Celia get each? How many bags of sweets will Anna buy in total?

Problem 12. Marie-Absa and Mariama want to earn some more pocket money, so they decide to do jobs around the house. Marie-Absa vacuums the carpets for 70p a time, and Mariama mows the lawn for £1.30 a time. If they do a different number of jobs each, they can end up with the same amount of money. What is the minimum number of jobs Marie-Absa and Mariama will each have to do in order to end up with the same amount of money? How much money will this be?

FUN WITH HCF, LCM, A PENTAGON AND ON HEXAGON

Find the Highest Common Factor and Lowest Common Multiple of these pairs of numbers.

32 , 44

14 , 35

15 , 50

18 , 24

16 , 20

M 4 , 352	P 5 , 150
I 4 , 80	R 6 , 72
T 8 , 352	E 7 , 70

Find the Highest Common Factor and Lowest Common Multiple of these pairs of numbers.

360 , 2100

462 , 378

540 , 1140

240 , 660

924 , 126

288 , 882

336 ,

	C 60 , 2640	R 60 , 12600
	E 60 , 7056	T 60 , 10260
O 42 , 4158	A 42 , 2772	F 42 , 7056

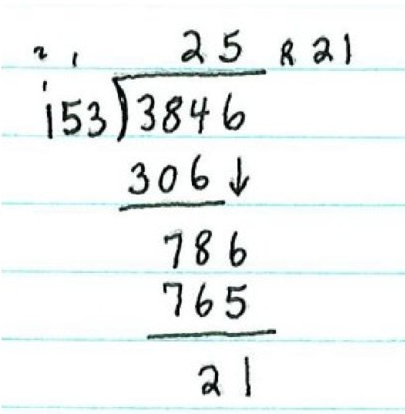
Academic Support Center
asc
Navigate Your Way

The Euclidean Algorithm

A METHOD FOR FINDING THE GREATEST COMMON DIVISOR FOR TWO LARGE NUMBERS

To be successful using this method you have got to know how to divide. If this is something that you have not done in a while and have forgotten or have never really mastered and have relied on the use of a calculator instead, you will first want to review the "Long Division" algorithm. Presented here is one example:

$3846 \div 153$



The Algorithm for Long Division

<u>Step 1</u>: Divide
<u>Step 2</u>: Multiply quotient by divisor
<u>Step 3</u>: Subtract result
<u>Step 4</u>: Bring down the next digit
<u>Step 5</u>: Repeat

When there are no more digits to bring down, the final difference is the remainder.

This can be rewritten in the form of what is known as the "Division Algorithm" (although it is not an algorithm):

$3846 = 153 \cdot 25 + 21$ *(dividend equals divisor times quotient plus remainder)*

(note that $0 \leq$ remainder $\leq$ divisor)

If you need more help with long division, go to You Tube and search "long division." Work through several examples and make sure you can successfully perform each example viewed on your own.

The greatest common divisor (gcd) of two integers, a and b, is the largest integer that divides evenly into both a and b.

We write gcd(a, b).

There are three methods for finding the greatest common factor.

<u>Method #1</u> The "easy" method: Inspection

This involves two numbers that, through experience, are easily grasped, such as 12 and 18.

Start with the smaller of the two numbers, 12. Does this divide into both numbers? (No, it does not divide evenly into 18.)

Since 12 didn't work, try the next largest integer that evenly divides 12 – by inspection that number is easily found to be 6. Does 6 also divide 18? Yes, therefore we are done – we have found the greatest common divisor and it is 6, hence, **gcd(12, 18) = 6**.

> **Now you try some:**
>
> Find the greatest common divisor of each by inspection.
>
> (a) gcd(24, 54)
>
> (b) gcd(18, 42)

<u>Method #2</u> Prime Factorization Method

The first step is to break each number into its prime factorization, then discern all the factors the two numbers have in common. Multiply these together. The result is the greatest common divisor.

Example 1: Find the gcd(168, 180)

$168 = 2^3 \cdot 3 \cdot 7 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$ $\qquad$ $180 = 2^2 \cdot 3^2 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

The common factors are $2 \cdot 2 \cdot 3 = 12$, therefore **gcf(168, 180) = 12**.

Example 2: Find the gcd(220, 1323)

$220 = 2^2 \cdot 5 \cdot 11$

$189 = 3^3 \cdot 7$

Observe that these two numbers have no common factors. So in this case the **gcd(220, 1323) = 1** and we say that the two integers are "*relatively prime.*

> **Now you try some:**
>
> Find the greatest common divisor of each by first finding the prime factorization of each number.
>
> (c) gcd(244, 354)
>
> (d) gcd(128, 423)

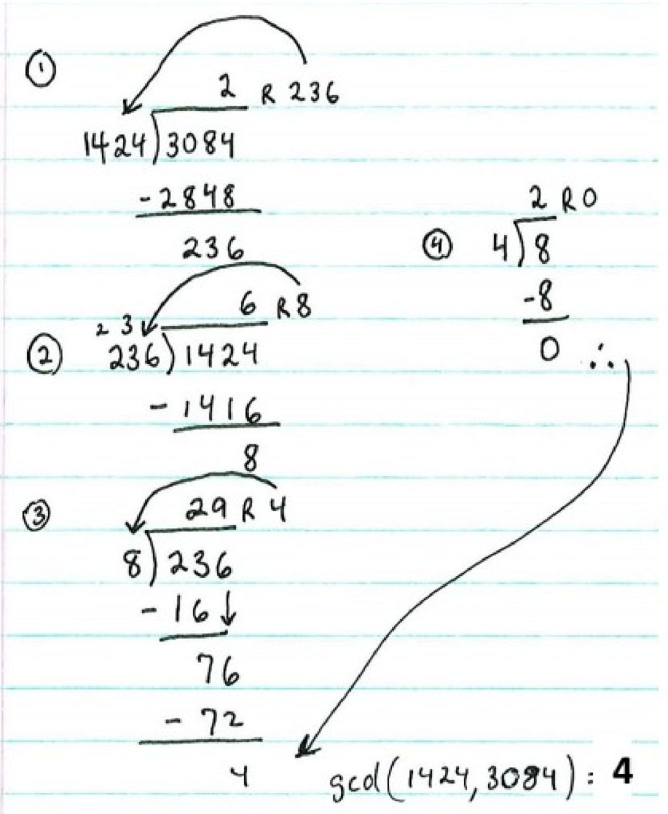
Method #3 The Euclidean Algorithm

This method asks you to perform successive division, first of the smaller of the two numbers into the larger, followed by the resulting remainder divided into the divisor of each division until the remainder is equal to zero. At that point, look at the remainder of the previous division – that will be the greatest common divisor.

Example:

Find the gcd(1424, 3084)

WORK SPACE:



PRESENTATION:

① $3084 = 1424 \cdot 2 + 236$

② $1424 = 236 \cdot 6 + 8$

③ $236 = 8 \cdot 29 + 4$

③ $8 = 4 \cdot 2 + 0$

Hence the **gcd(1424, 3084) = 4**

Note: Gabriel Lamé (1795-1870) found that the number of steps required in the Euclidean Algorithm is – <u>at most</u> – 5 times the number of digits in the smaller number.

Now you try some:

Find the greatest common divisor of each by first finding the prime factorization of each number.

(e) gcd(2415, 3289)

(f) gcd(4278, 8602)

(g) gcd(406, 555)

Why does the Euclidean Algorithm work?

The example used to find the **gcd(1424, 3084)** will be used to provide an idea as to why the Euclidean Algorithm works. Let ***d*** represent the greatest common divisor. Since this number represents the largest divisor that evenly divides both numbers, it is obvious that ***d*****|1424** and ***d*****|3084**. Hence ***d*****|3084 –1424** in the same way that a numerator of two or more terms is divisible by a factor in the denominator if and only if that factor divides evenly into each individual term in the numerator. Furthermore ***d*****|3084 – 2(1424)** or, simplifying the left side, ***d*****|236**. Consequently we note that ***d*** divides the remainder resulting from the division of the two numbers for which the greatest common factor is being sought. This corresponds to the first long division calculated in the example above.

The problem has now been reduced to a smaller one. We have that ***d*****|1424** and ***d*****|236**. Hence ***d*****|1424 – 236**, or better yet, ***d*****|1424 – 6(236)** which when simplified reveals that ***d*****|8**. At this point we know that ***d*** must be one of the following values: 1, 2, 4, or 8. Note that 8 is the remainder resulting from the division of the divisor and remainder found in the original division, so it will not be a divisor of both. So we will take the divisor and remainder from the second division to reduce the problem to yet an even smaller one.

Now we know that ***d*****|236** and ***d*****|8**, so ***d*****|236 – 8** or ***d*****|236 – 29(8)**, which leaves us, after calculation, with the fact that ***d*****|4**. This, of course, corresponds to the third long division performed above.

One last long division reduces the problem one more level – the final level. We have that ***d*****|8** and ***d*****|4**, hence ***d*****|8-2(4)** or ***d*****|0**. We can go no further. ***d*****|0** does not provide us with any useful information. (Why not?) But ***d*****|4**, where **4** is seen to be the remainder of the last long division, tells us that ***d*** can be 1, 2, or 4. The largest amongst these is **4** – so ***d*** must be equal to **4** and we are done. We have thus discovered that ***d*** which equals the **gcd(1424, 2084)** is equal to **4**.

Source: Rochester Institute of Technology, NY, USA. http://www.rit.edu/~w-asc/documents/services/resources/handouts/DM%20-%206%20Euclidean%20Algorithm.pdf

TEACHER'S CORNER

♠ Solution to problem 11.

HCF and LCM star matching

Instructions

Students should cut out all of the shapes on the page, calculate the HCF and LCM of the pairs of numbers in the pentagon/hexagon, and match the correct triangles round the edges. The resulting star shape will spell out a word.

Teaching notes

This resource contains two matching activities on HCF and LCM. The hexagon set is harder than the pentagon set as it involves higher numbers and uses only two HCFs.

For a shorter activity, or to differentiate between different students, you could separate the hexagon and pentagon sets.

For a longer activity, you could cut the shapes out before the lesson, so students don't know whether each triangle belongs in the hexagon or pentagon set. To save time, cut the shapes roughly for students to finish themselves.

Both sets contain one 'red herring', making it harder for students to guess the hidden word.

You could adapt the activity as a starter to introduce a different topic, by changing the hidden words to suit the learning objective. E.g. 'shape', 'metre', 'BIDMAS', 'divide', etc. All text in the shapes was made using WordArt.

Answers

Pentagon set

(hidden word read clockwise)

Pentagon	15 , 50	18 , 24	16 , 20	32 , 44	14 , 35
Triangle	5 , 150	6 , 72	4 , 80	4 , 352	7 , 70
	P	**R**	**I**	**M**	**E**

Hexagon set

(hidden word read anticlockwise)

Hexagon	336 , 882	924 , 126	240 , 660	540 , 1140	462 , 378	360 , 2100
Triangle	42 , 7056	42 , 2772	60 , 2640	60 , 10260	42 , 4158	60 , 12600
	F	**A**	**C**	**T**	**O**	**R**

 21451 Page 2 of 2

♠ Solutions for "The Euclidean Algorithm" problems.

Now you try some: Answers		
(a) gcd(24, 54) = 6 (b) gcd(18, 42) = 6	(c) gcd(244, 354) = 2 (d) gcd(128, 423) = 1	(e) gcd(2415, 3289) = 23 (f) gcd(4278, 8602) = 46 (g) gcd(406, 555) = 1

LESSON PLANS FOR ARITHMETICS

- ☐ Intro: *(Doc: Sujet bac 2012-04: The amazing number 6174)*
- ☐ Intro: *(Doc: Sujet bac 2012 :Division by 0)*
- ☐ Intro: *(Doc: Numberphile , 3 is everywhere)*
- ☐ Prime numbers including Erathostenes' sieve *(Doc: Prime numbers, Ac Bordeaux)*
- ☐ You may take a look at : http://en.wikipedia.org/wiki/Sieve_of_Eratosthenes to see the sieve of Eratosthenes in action!
- ☐ Why 1 not a prime number + Fundamental Thm of Arithmetic, *(Doc: Numberphile, 1 not a prime number + Questions LHG).*
- ☐ Fundamental Thm of Arithmetic http://www.mathsisfun.com/numbers/fundamental-theorem-arithmetic.html (déja vu dans la video “Why 1 not a prime”, mais pour l'avoir sous forme écrite)
- ☐ Bitesize GCSE : Find the prime factors using a tree http://www.bbc.co.uk/schools/gcsebitesize/maths/number/primefactorsrevl.shtml
- ☐ Fun with HCF and LCM, create word from pentagon and hexagon and guess the hidden word http://www.teachitmaths.co.uk/?resource=21451
- ☐ The Euclidean Algorithm http://www.rit.edu/~w-asc/documents/services/resources/handouts/DM%20-%206%20Euclidean%20Algorithm.pdf
- ☐ There is an infinite number of primes *(Doc: Numberphile , infinite primes+ questions LHG)*
- ☐ ? Cryptography *(Doc: Plying spy Emilangue)*
- ☐ ? Sujets de bac *(Doc: Sujets ????)*

TABLE OF CONTENTS