Functions, Part I | DNL 1ère

Mathematical topic: coordinates, straight lines, linear functions, functions, quadratic functions.[NO derivatives]

Vocabulary: x-axis, y-axis, -coordinate, y-coordinate, x -intercept, linear functions, "slope of a line, f(x), image, pre-image", domain and range of a function...

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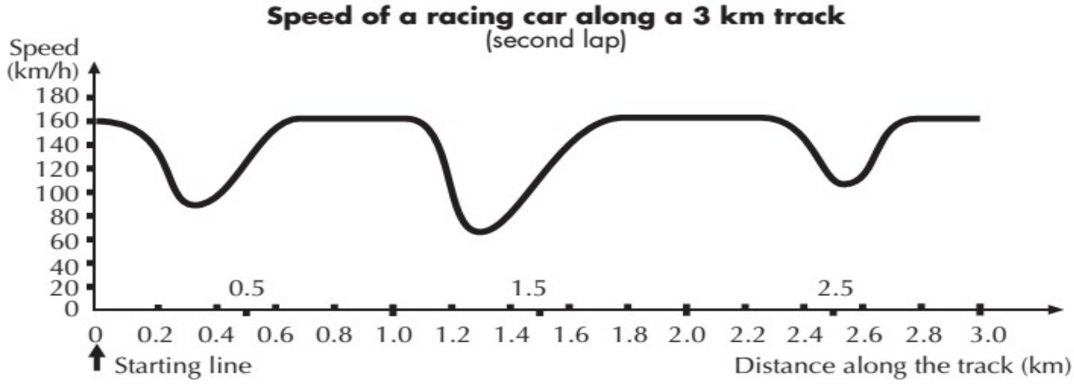
◯ Problem 1. Speed of a racing car [Source : PISA]

This graph shows how the speed of a racing car varies along a flat 3 kilometre track during its second lap.



1) *What is the approximate distance from the starting line to the beginning of the longest straight section of the track?*

A *0.5 km* B *1.5 km* C *2.3 km* D *2.6 km*

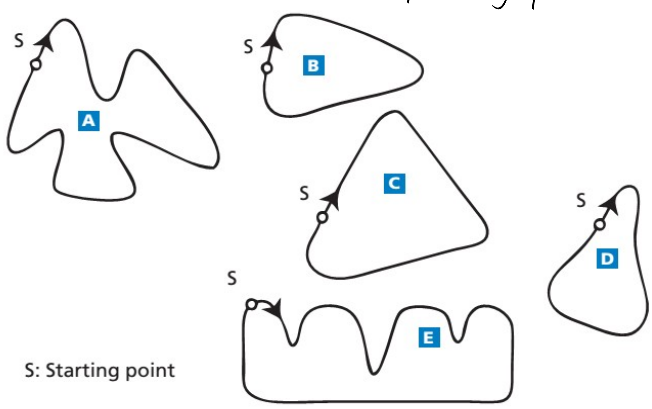
2) *Where was the lowest speed recorded during the second lap?*

A *at the starting line.* B *at about 0.8 km.* C *at about 1.3 km.* D *halfway around the track.*

3) *What can you say about the speed of the car between the 2.6 km and 2.8 km marks?*

A *The speed of the car remains constant.* B *The speed of the car is increasing.* C *The speed of the car is decreasing.* D *The speed of the car cannot be determined from the graph.*

4) *Here are pictures of five tracks:*
Along which one of these tracks was the car driven to produce the speed graph shown earlier? Explain your answer.



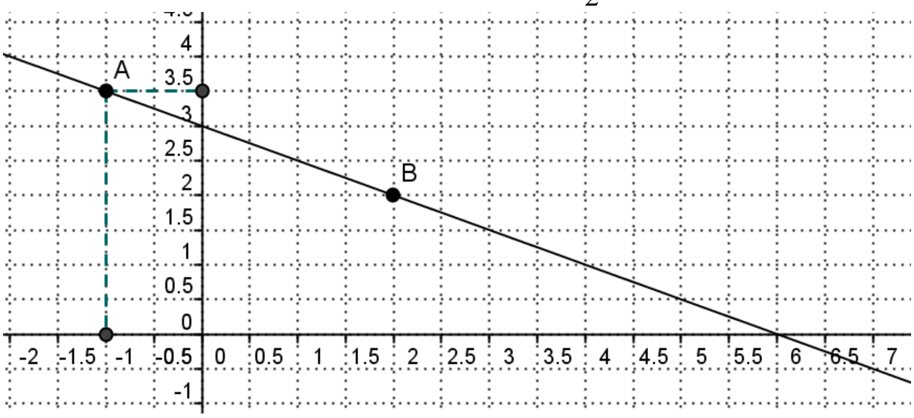
I. Rectangular coordinate system – Straight lines- Linear functions

Useful words and concepts (You will need them so just learn them!)

- The ***x-axis*** is the horizontal real number line. The ***y-axis*** is the vertical real number line.
- Every point on the coordinate plane is described by a ***coordinate pair*** (x, y). The first of these numbers is called the ***x-coordinate*** or ***abscissa*** and the second one is ***y-coordinate or ordinate***.
  Pronunciation : /ˈæk sɪs/ /ˌhɔr əˈzɒn tl/ /ˈvɜr tɪ kəl/ /koʊˈɔr dn ɪt/ /pɛər/ /æbˈsɪs ə/ /ˈɔr dnˌɪt/
- The point where the graph of a function graph intersects the y-axis is called the ***y-intercept*** of the graph. The word *y-intercept* is also used for the y-coordinate of this point. Similarly, the point(s), if any, where the graph of a function intersects the x-axis are called the *x-intercepts* of the graph. The word x-intercept is also used for the *x-coordinate* of these points.
- If a function is of the form $f(x)=mx+p$, then its graph is the ***straight line*** with equation $y=mx+p$, m is its ***slope*** and p is its ***y-intercept***. Such a function is called a ***linear*** function.
  Pronunciation : /streɪt laɪn/ /ɪˈkweɪ ʒən, -ʃən/ /sloʊp/ /ˈɪn tərˌsɛpt/ /ˈlɪn i ər/
  The French for « linear function » is. . . . . , NOT. . . . .
- The slope m of a non vertical line that goes through the points $A(x_A, y_A)$ and $B(x_B, y_B)$ is given by the formula $m=\frac{y_B-y_A}{x_B-x_A}=\frac{y_A-y_B}{x_A-x_B}=\frac{rise}{run}$.
- The slope of a line is a measure of its *steepness*: Lines with *positive slope* slant upward towards the right whereas lines with *negative slope* slant downward towards the right. The steepest lines are the ones for which the absolute value of the slope is largest. A horizontal line has slope zero. The slope of a vertical line is not defined.
- In order to ***sketch the graph*** of a linear function, you just need to ***plot*** two points **on** the coordinate plane. (It takes two points to define a line!).
- In order to construct the ***table*** of values for a function, ***substitute*** a few values ***for*** x, ***compute*** the corresponding y-values and *record* the resulting pair in the table.
  Remember that your calculator can do this for you!

○ Problem 2. A linear function to start with:

Let f be the function defined on ℝ by $f(x)=-\frac{1}{2}x+3$. Its graph is the straight line shown below.



"Functions will haunt you for the rest of your life so, unless you face your fears now, you will end up sleeping with the lights on for the rest of your mathematical career."
Michael Kelley

1) Coordinates of A and B:
The x-coordinate of A is –1 and its y-coordinate is 3.5 which we denote by A(–1, 3.5).
Fill in the blanks: *The x-coordinate of B is . . . . . . and its y-coordinate is . . . . . . which we denote by. . . . .*

2) Determine the y-intercept of this straight line.

3) Determine the x-intercept of this line, which indicates where the graph intersects the x-axis.

4) Determine the slope of this line.

5) Determine the image of –2 by f.

○ Problem 3.

Let f be a linear function whose graph goes through the points $A(3,1)$ and $B(6,-5)$.

1) Find an expression for f in the point-slope form.

2) Find an expression for f in slope-intercept form.

Useful words

- $y = m(x - x_1) + y_1$ is the equation of a straight line in ***point-slope form***. Such a straight line has slope m and goes through the point (x_1, y_1).
- $y = mx + p$ is the equation of a straight line in ***slope-intercept form***. m is its ***slope*** and p is its ***y-intercept***.

○ Problem 4.

1) Write the area of a square as a function of its perimeter. Is this a linear function?
2) Write the perimeter of a square as a function of its area. Is this a linear function?
3) Write the circumference (= perimeter) of a circle as a function of its radius. Is this a linear function?
4) Write the area of a disk as a function of its diameter. Is this a linear function?

○ Problem 5. Air temperature. Stewart p 26 with solution

1) As dry air moves upward, it expands and cools. If the ground temperature is $20^\circ C$, and the temperature at a height of 1 km is $10^\circ C$, express the temperature T (in $^\circ C$) as a function of the height h (in kilometres), assuming that a linear model is appropriate.
2) Draw the graph of the function in part (1).
3) What is the slope equal to (specify the unit)? What does the slope represent? What does the y-intercept represent?
4) What is the temperature at the height 2.5 km?
5) At what height is the temperature equal to $-24^\circ C$?

○ Problem 6. Car pooling. Calculus for business and Economics p 46

To encourage motorists to form carpools, the transit authority in a certain metropolitan area has been offering a special reduced rate at toll bridges for vehicles containing four or more persons. When the program began 30 days ago, 157 vehicles qualified for the reduced rate during the morning rush hour. Since then, the number of vehicles qualifying has been increasing at a constant rate, and today 247 vehicles qualified.

Useful words

- encourage /ɛnˈkɜrɪdʒ
- motorist /ˈmoʊtərɪst/
- carpool /ˈkɑrˌpul/
- vehicle /ˈviɪkəl/
- reduce /rɪˈdus, -ˈdyus/

1) a) Be ready to describe (orally) with your own words what the transit authority is trying to achieve and how. Make sure you explain the words "toll", " rush hour" and "car pool".
b) Do you know of a town with such a policy?

2) a) Express the number of vehicles qualifying each morning for the reduced rate as a function of time and draw the graph.
b) If the trend continues, how many vehicles will qualify during the morning rush hour 14 days from now?

○ Problem 7. Driving uphill and downhill

1) You are driving on a road that has a 6% uphill grade. A 6 % grade means that the altitude changes by 6 feet for each 100 feet of horizontal distance. Approximate the amount of vertical change in your position if you drive 360 feet (which is the distance to the top of the mountain). By the way, how many meters is that?

2) After enjoying the view, you decide to go downhill and that's when you discover that you need to take a road that has a 13% downhill grade!

a) Approximate the amount of vertical change in your position if you drive 40 meters downhill on this road.
b) Will you be back to your original height?
c) If not, how many meters should you be driving uphill or downhill on this road in order to back to your original height?

Problem 8. Alternative-fueled vehicles

The number V (in thousands) of alternative-fueled vehicles in the United States increased in a linear pattern from 1995 to 1999. Then in 2000, the number of such vehicles took a jump and, until 2002, increased in a different linear pattern.
These two patterns can be approximated by the function

$$V(t)=\begin{cases}18.08t+155.3 & \text{if } 5\le t\le 9\\ 32.20t+10.2 & \text{if } 10\le t\le 12\end{cases}$$

where t represents the year, with $t=5$ corresponding to 1995.

Useful words

- A ***piecewise defined function*** is defined by different formulas in different parts of its domain. /ˈpis,waɪz/
- vehicle /ˈviɪkəl/
- alternative /ɔlˈtɜrnnətɪv/
- pattern /ˈpætəərn/

1) Use this function to approximate the number of alternative-fueled vehicles in the United States for each year from 1995. Report your findings in a table. Sketch the graph of V.
You may use your calculator or a spreadsheet program (Open Office Calc for instance) to produce these numbers.

2) By how many percent did the number of alternative-fueled vehicles in the United States increase each year between 1995 and 2002? You may write a program on your calculator or a formula in a spreadsheet (in Open Office Calc for instance) to determine these percentages.

II. Functions, general case

Problem 9. Based on a Khan Academy video : https://www.khanacademy.org/math/algebra/algebra-functions/relationships_functions/v/what-is-a-function

Watch the video up to 2:00 and fill in the blanks.

a) Based on an input, the function will produce a given . . . .

b) The letter usually used for the input value of a function is . . . .

c) In order to compute $f(2)$, wherever we see . . . . . . we replace it with . . . . .

d) Let f be the function described in the video.
$f(6)=$ . . . . and $f(-3)=$ . . . . . .

e) Let f be the function described in the video. The video doesn't specify its domain but it is implied that the domain is . . . . . .

Useful words

- 0, 1, 2, 3, 4… are ***whole*** [hoʊl] numbers whereas -3 ; 1.6 and 2/3 are not.
- The *whole* numbers together with their *opposites* form the set of ***integers*** [ˈɪntɪdʒər]. For example, -3 and $+17$ are integers whereas 1.6, $-\sqrt{7}$ and 4/3 are not.
- 0, 2, 4, 6... are ***even*** numbers.
- 1, 3, 5, 7... are ***odd*** numbers.

Problem 10. _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

Let f be the function defined on $\mathbb{R}$ by $\begin{cases} f(x)=x & \text{if } x\ge 0,\\ f(x)=-x & \text{if } x<0.\end{cases}$

1) Determine $f(-8.4)$, $f(\sqrt{2})$, $f(-\sqrt{5})$ and $f(1-\sqrt{2})$.

2) a) What is the sign of $f(x)$ depending on x ?

b) Sketch the graph of this function.

c) Do you recognize this function? If so, write its name as the title for this problem.

Problem 11.

Sketch a possible graph of a function f which satisfies *all* of the following conditions:

- Its domain is $\mathbb{R}\setminus\{3\}$.
- Its x-intercepts are 1 and 4 and its y-intercept is 1.5.
- f is increasing on]3, 5].
- f is decreasing on $]-\infty,3[$ and on $[5,+\infty[$.
- $f(5) = 2$ and $f(7) = 1$

You may first record all the information in a variation chart.

Problem 12.

Let f be the function defined on $\mathbb{R}$ by $f(x)=(x+1)^2-2$.

1) What is the *domain* of f ?

2) a) Determine $f(5), f(-5), f(0)$ and $f\left(\frac{2}{3}\right)$.

b) Determine the image of $3-2\sqrt{3}$ by *f*. *Simplify the expression as much as possible.*

3) a) Determine (if possible) the pre-image(s) of 7 by *f*.

b) Determine (if possible) the pre-image(s) of 1 by *f*.

c) Determine (if possible) the pre-image(s) of -5 by *f*.

d) Determine (if possible) the pre-image(s) of -2 by *f*.

e) *Challenge question*: Let k be a real number. Determine the number of pre-image(s) of k by *f* depending on k. *The expected answer is of the form "For k<..., k has ... (1 or 2 or ...) pre-images; ...etc"*

4) What is the *range* of f ?

Useful words

- The ***domain*** of a function is the set of all x for which the function is defined. In particular, it excludes all the x-values that result in a division by 0 or in taking the square roots of negative numbers.
- The ***image*** of x by f is $f(x)$.
- A ***pre-image*** of a number y by f is a number x such that $y=f(x)$.
- The ***codomain*** of a function is a set containing the function's outputs, whereas the ***range*** is the part of the codomain which consists only of the function's outputs. In other words, the ***range of f*** is the set of all possible values of $f(x)$ as x varies throughout the domain.

Need help with the math or the English? Go to the mathematoques website http://mathematoques.weebly.com and it will point you to videos that explain how to compute the image of a number by a function and much more !

Problem 13.

1) Let f be the function defined on $\mathbb{R}$ by $f(x)=\sqrt{4-3x}$.

2) Determine (if possible) $f(-4)$, $f\left(\frac{88}{75}\right)$, $f(0)$ and $f(4)$.

3) Determine the image of -8 by *f*.

4) Determine (if possible) the pre-image(s) of –8 by *f*.

5) Determine (if possible) the pre-image(s) of 0 by *f*.

6) Determine (if possible) the pre-image(s) of 2 by *f*.

7) What is the *domain* of *f*?

8) What is the *range* of *f*?

Problem 14. Programming a function on your calculator or Algobox

Let f be the function defined on $\mathbb{R}$ by $\begin{cases} f(x)=2x-3 & \text{if } x\geq 1, \\ f(x)=-x^2 & \text{if } x<1. \end{cases}$

Our goal is to fill in the following table. *[Don't do it now, keep reading!]*

x	–3	–2	–1	–1/2	–1/4	0	1/4	1/2	1	2	3	4	5
f(*x*)													

1) Compute by hand $f(3), f(-1), f(1)$ and $f(\tilde{}2)$ and report these values in the table above.

2) On your notebook , write a program an algorithm that, given *x*, computes *f*(*x*), using pseudo code.

3) Enter the program in your calculator or in Algobox.

4) Fill in the table with the values of the function.

5) Sketch the graph of this function by plotting the points from the table. Could you have predicted the shape of the graph?

6) Determine (if possible) the pre-image(s) of 3 by *f*.

7) Determine (if possible) the pre-image(s) of $-\frac{1}{9}$ by *f*.

8) What is the range of *f*?

Problem 15. Let f be the function defined on the set of whole numbers by $f(x)=\begin{cases} x^2+1 & \text{if } x \text{ is even,} \\ x^2-1 & \text{if } x \text{ is odd.}\end{cases}$

Useful words
Given any two integers, if both are odd or both even they are said to have the same ***parity*** /ˈpærɪti/; if one is odd and one even they have different parity.

1) Determine $f(3)\,f(6), f(5)$ and $f(10)$.

2) Determine a general formula for $f(2k)$ and a general formula for $f(2k+1)$ where k is a whole number.

3) a) Determine $f(f(1))$ and $f(f(2))$.

b) Determine a general formula for $f(f(x))$ and specify whether it is even or odd depending on whether x is even or odd.

4) Solve (if possible) the equation $f(x)=170$.

5) Solve (if possible) the equation $f(f(x))=65$.

6) Solve (if possible) the equation $f(f(x))=64$.

7) Solve (if possible) the equation $f(x)=-64$.

III. Quadratic functions and parabolas

Useful words and concepts

- If the expression of function can be written $f(x)=ax^2+bx+c$ (with $a\neq 0$) then the function is called a ***quadratic*** function. The graph of such a function is always a ***parabola***. The parabola opens upward if $a>0$ and it opens downward if $a<0$. a is called the ***leading coefficient***.
- All parabolas are *symmetric with respect to* a line called the ***axis of symmetry***. A parabola intersects its axis of symmetry at a point called the ***vertex*** of the parabola.
- $y=a(x-x_1)^2+y_1$ is the equation of a parabola in ***standard form***, the vertex being the point $V(x_1;y_1)$. The standard form is also called the ***canonical form*** or the ***vertex form***.
- It is always possible to transform the ***expanded form*** $f(x)=ax^2+bx+c$ into the standard form, and $f(x)=a(x-x_1)^2+y_1$ vice versa, as we shall see in the examples. Rewriting *f*(*x*) in the vertex form enables us to find the coordinates of the vertex.

Exercise 16. Optimal selling price

A manufacturer can produce bookcases at a cost of \$80 apiece. Sales figures indicate that if bookcases are sold for *x* dollars apiece, approximately 150–*x* will be sold each month.

1) Express the manufacturer's monthly profit P as a function of the selling price *x*, draw a graph and estimate the optimal selling price.

2) Can you determine algebraically the exact value of the optimal selling price?

Exercise 17. Rewriting a quadratic Function in vertex form in order to find its extremum.

Our goal here is to find the rectangle with maximal area among the rectangles with perimeter equal to 8 feet.

1) Write the area $\mathcal{A}(x)$ of a rectangles with perimeter 8 feet as a function of the length *x* of one of its sides.

2) *Graphical approach*: Use your calculator to sketch the graph of $\mathcal{A}(x)$ and make a conjecture about the dimensions that yields a maximum area.

The graphical approach will never produce an exact answer so we try to improve on it by using an algebraic approach.

3) *Algebraic approach:*

a) Is $\mathcal{A}(x)$ a quadratic function? If so, rewrite it in vertex form.

b) Now, solve the exercise.

You know that two points determine a line. This means that if you are given any two points in the plane, then there is one and only one line that contains both points. A similar statement can be made about points and quadratic functions.
Given three points in the plane that have different first coordinates and do not lie on a line, there is exactly one quadratic function f whose graph contains all three points. The applet below illustrates this fact. The graph contains three points and a parabola that goes through all three. The corresponding function is shown in the text box below the graph. If you drag any of the points, then the function and parabola are updated.

❍ Problem 18.

Find the vertex of the graph of f(x) = (x + 9)(x - 5).
Since the formula for f is factored, it is easy to find the zeros: -9 and 5.
The average of the zeros is (-9 + 5)/2 = -4/2 = -2. So, the line of symmetry is x = -2 and the first coordinate of the vertex is -2.
The second coordinate of the vertex is f(-2) = (-2 + 9)(-2 - 5) = 7*(-7) = -49.
Therefore, the vertex of the graph of f is (-2, -49).

❍ Problem 19.

A rancher has 600 meters of fence to enclose a rectangular corral with another fence dividing it in the middle as in the diagram below.

As indicated in the diagram, the four horizontal sections of fence will each be x meters long and the three vertical sections will each be y meters long.
The rancher's goal is to use all of the fence and **enclose the largest possible area**.
The two rectangles each have area xy, so we have
total area: A = 2xy.
There is not much we can do with the quantity A while it is expressed as a product of two variables. However, the fact that we have only 1200 meters of fence available leads to an equation that x and y must satisfy.
3y + 4x = 1200.
3y = 1200 - 4x.
y = 400 - 4x/3.
We now have y expressed as a function of x, and we can *substitute this expression for y* in the formula for total area A.
A = 2xy = 2x (400 -4x/3).
We need to find the value of x that makes A as large as possible. A is a quadratic function of x, and the graph opens downward, so the highest point on the graph of A is the vertex. Since A is factored, the easiest way to find the vertex is to find the x-intercepts and average.
2x (400 -4x/3) = 0.
2x = 0 or 400 -4x/3 = 0.
x = 0 or 400 = 4x/3.
x = 0 or 1200 = 4x.
x = 0 or 300 = x.
Therefore, the line of symmetry of the graph of A is x = 150, the average of 0 and 300.
Now that we know the value of x corresponding to the largest area, we can find the value of y by going back to the equation relating x and y.
y = 400 - 4x/3 = 400 -4(150)/3 = 200.